Highest weight representations of the Virasoro algebra

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Abstract. We consider representations of the Virasoro algebra, a one-dimensional central extension of the Lie algebra of vectorfields on the unit circle. Positive-energy, highest weight and Verma representations are defined and investigated. The Shapovalov form is introduced, and we study Kac formula for its determinant and some consequences for unitarity and degeneracy of irreducible highest weight representations. In the last section we realize the centerless Ramond algebra as a super Lie algebra of superderivations.

Contents

1 Introduction

In this second part of the master thesis we review some of the representation theory for the Virasoro algebra. It is the unique nontrivial one-dimensional central extension of the Witt algebra, which is the Lie algebra of all vectorfields on the unit circle. More specifically we will study highest weight representations, which is an important class of representations. Shapovalov ([5]) defined a Hermitian form on any highest weight representation. This in particular induces a nondegenerate form on the irreducible quotient. Thus properties of irreducible highest weight representations can be studied in terms of this form. In [2], [3] Kac gave a formula for the determinant of the Shapovalov form. The formula was proved by Feigin and Fuchs in [1].

In Section 2 we introduce some notation that will be used throughout the article. The Witt algebra is defined algebraically as the Lie algebra of all derivations of Laurent polynomials. We show that it has a unique nontrivial one-dimensional central extension, namely the Virasoro algebra. We define highest weight, positive energy, and Verma representations in Section 3. Conditions for an irreducible highest weight representation to be degenerate or unitary are considered in Section 4. We also provide some lemmas to support the main theorem (Theorem 28), the Kac determinant formula, although we do not give a complete proof. Finally, in Section 5 we consider a supersymmetric extension of the Witt algebra, and we show that it has a representation as superderivations on $\mathbb{C}[t, t^{-1}, \epsilon \mid \epsilon^2 = 0]$. Superderivations are special cases of σ-derivations, as described in the first part of the master thesis.

2 Definitions and notations

For a Lie algebra $\mathfrak{g}$, let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra.

Definition 1 (Extension)**.** Let $\mathfrak{g}$ and I be Lie algebras. An *extension* $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by I is a short exact sequence

$$0 \longrightarrow I \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

of Lie algebras. The extension is *central* if the image of I is contained in the center of $\widetilde{\mathfrak{g}}$, and *one-dimensional* if I is.

Note that $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}\oplus I$ as linear spaces. Given two Lie algebras $\mathfrak{g}$ and I, one may always give $\mathfrak{g}\oplus I$ a Lie algebra structure by defining $[x+a, y+b]_{\mathfrak{g}\oplus I} = [x,y]_{\mathfrak{g}} + [a,b]_I$ for $x, y \in \mathfrak{g}, a, b \in I$. This extension is considered to be trivial.

Definition 2 (Antilinear anti-involution)**.** An antilinear anti-involution ω on a complex algebra A is a map $A \to A$ such that

$$\omega(\lambda x + \mu y) = \overline{\lambda}\omega(x) + \overline{\mu}\omega(y) \quad \text{for } \lambda, \mu \in \mathbb{C}, x, y \in A \tag{1}$$

and

$$\omega(xy) = \omega(y)\omega(x) \quad \text{for } x, y \in A, \tag{2}$$

$$\omega(\omega(x)) = x \quad \text{for } x \in A. \tag{3}$$

Definition 3 (Unitary representation)**.** Let $\mathfrak{g}$ be a Lie algebra with an antilinear anti-involution $\omega : \mathfrak{g} \to \mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ in a linear space V equipped with an Hermitian form $\langle \cdot, \cdot \rangle$. The form $\langle \cdot, \cdot \rangle$ is called *contravariant* with respect to ω if

$$\langle \pi(x)u, v \rangle = \langle u, \pi(\omega(x))v \rangle \quad \text{for all } x \in \mathfrak{g}, u, v \in V.$$

The representation π is said to be *unitary* if in addition $\langle v, v \rangle > 0$ for all nonzero $v \in V$.

Remark 1. If only one representation is considered, we will often use module notation and write xu for $\pi(x)u$ whenever it is convenient to do so.

The following Lemma will be used a number of times.

Lemma 1. *Let V be a representation of a Lie algebra $\mathfrak{g}$ which decomposes as a direct sum of eigenspaces of a finite dimensional commutative subalgebra $\mathfrak{h}$:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \tag{4}$$

where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$, and $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h}$. Then every subrepresentation U of V respects this decomposition in the sence that

$$U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda).$$

Proof. Any $v \in V$ can be written in the form $v = \sum_{j=1}^m w_j$, where $w_j \in V_{\lambda_j}$ according to (4). Since $\lambda_i \neq \lambda_j$ for $i \neq j$ there is an $h \in \mathfrak{h}$ such that $\lambda_i(h) \neq \lambda_j(h)$ for $i \neq j$. Now if $v \in U$, then

$$\begin{array}{rcccccccc}
v & = & w_1 & + & w_2 & + & \ldots & + & w_m \\
h(v) & = & \lambda_1(h)w_1 & + & \lambda_2(h)w_2 & + & \ldots & + & \lambda_m(h)w_m \\
& \vdots & & & & & & & \\
h^{m-1}(v) & = & \lambda_1(h)^{m-1}w_1 & + & \lambda_2(h)^{m-1}w_2 & + & \ldots & + & \lambda_m(h)^{m-1}w_m
\end{array}$$

The coefficient matrix in the right hand side is a Vandermonde matrix, and thus invertible. Therefore each w_j is a linear combination of vectors of the form $h^i(v)$, all of which lies in U, since $v \in U$ and U is a representation of $\mathfrak{g}$. Thus each $w_j \in U \cap V_{\lambda_j}$ and the proof is finished. $\square$

2.1 The Witt algebra

The Witt algebra $\mathfrak{d}$ can be defined as the complex Lie algebra of derivations of the algebra $\mathbb{C}[t, t^{-1}]$ of complex Laurent polynomials. Explicitly,

$$\mathbb{C}[t, t^{-1}] = \{\sum_{k \in \mathbb{Z}} a_k t^k \;\big|\; a_k \in \mathbb{C}, \text{only finitely many nonzero}\}$$

and

$$\mathfrak{d} = \{D : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}] \;\big|\; D \text{ is linear and } D(pq) = D(p)q + pD(q)\} \tag{5}$$

with the usual Lie bracket: $[D, E] = DE - ED$. One can check that $\mathfrak{d}$ is closed under this product. The following proposition reveals the structure of $\mathfrak{d}$.

Proposition 2. *Consider the elements d_n of $\mathfrak{d}$ defined by*

$$d_n = -t^{n+1} \frac{d}{dt} \quad \text{for } n \in \mathbb{Z}.$$

Then

$$\mathfrak{d} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n \tag{6}$$

and

$$[d_m, d_n] = (m - n) d_{m+n} \quad \text{for } m, n \in \mathbb{Z}. \tag{7}$$

Proof. Clearly $\mathfrak{d} \supseteq \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n$. To show the reverse inclusion, let $D \in \mathfrak{d}$ be arbitrary. Then, using (5), i.e. that D is a derivation of $\mathbb{C}[t, t^{-1}]$, we obtain

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Hence $D(1) = 0$, which implies that

$$0 = D(t \cdot t^{-1}) = D(t) \cdot t^{-1} + t \cdot D(t^{-1}),$$

which shows that

$$D(t^{-1}) = D(t) \cdot (-t^{-2}). \tag{8}$$

Now define the element $E \in \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n$ by

$$E = D(t) \frac{d}{dt},$$

and note that $E(t) = D(t)$. Note further that $E(t^{-1}) = D(t) \cdot (-t^{-2})$ and thus, by (8), that the derivations E and D coincide on the other generator t^{-1} of $\mathbb{C}[t, t^{-1}]$ also. Using the easily proved fact that a derivation of an algebra is uniquely determined by the value on the generators of the algebra, we conclude that $D = E$. Therefore $\mathfrak{d} \subseteq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$ and the proof of (6) is finished.

We now show the relation (7). For any $p(t) \in \mathbb{C}[t, t^{-1}]$, we have

$$\begin{aligned}(d_m d_n)(p(t)) &= d_m(-t^{n+1} \cdot p'(t)) = \\ &= d_m(-t^{n+1}) \cdot p'(t) + (-t^{n+1}) \cdot d_m(p'(t)) = \\ &= -t^{m+1} \cdot (-(n+1))t^n \cdot p'(t) + (-t^{n+1})(-t^{m+1})p''(t) = \\ &= (n+1)t^{m+n+1} \cdot p'(t) + t^{m+n+2}p''(t).\end{aligned}$$

The second of these terms is symmetric in m and n, and therefore vanishes when we take the commutator, yielding

$$[d_m, d_n](p(t)) = \big((n+1) - (m+1)\big)t^{m+n+1}p'(t) = (m-n) \cdot d_{m+n}(p(t)),$$

as was to be shown. $\square$

Remark 2. Note that the commutation relation (7) shows that $\mathfrak{d}$ is $\mathbb{Z}$-graded as a Lie algebra with the grading (6).

2.2 Existence and uniqueness of Vir

Theorem 3. *The Witt algebra $\mathfrak{d}$ has a unique nontrivial one-dimensional central extension $\widetilde{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{C}\overline{c}$, up to isomorphism of Lie algebras. This extension has a basis $\{c\} \cup \{d_n \mid n \in \mathbb{Z}\}$, where $c \in \mathbb{C}\overline{c}$, such that the following commutation relations are satisfied:*

$$[c, d_n] = 0 \qquad \text{for } n \in \mathbb{Z}, \tag{9}$$

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}c \qquad \text{for } m, n \in \mathbb{Z}. \tag{10}$$

The extension $\widetilde{\mathfrak{d}}$ is called the *Virasoro algebra*, and is denoted by Vir.

Proof. We first prove uniqueness. Suppose $\widetilde{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{C}\overline{c}$ is a nontrivial one-dimensional central extension of $\mathfrak{d}$. Let $\overline{d}_n, n \in \mathbb{Z}$ denote the standard basis elements of $\mathfrak{d}$, then we have

$$\begin{aligned}[\overline{d}_m, \overline{d}_n] &= (m-n)\overline{d}_{m+n} + a(m,n)\overline{c} \\ [\overline{c}, \overline{d}_n] &= 0\end{aligned} \tag{11}$$

for $m, n \in \mathbb{Z}$, where $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is some function. Note that we must have $a(m,n) = -a(n,m)$ because $\widetilde{\mathfrak{d}}$ is a Lie algebra and thus has an anti-symmetric product:

$$0 = [\overline{d}_m, \overline{d}_n] + [\overline{d}_n, \overline{d}_m] = (m - n + n - m)\overline{d}_0 + (a(m,n) + a(n,m))\overline{c}.$$

Define new elements

$$d'_n = \begin{cases} \overline{d}_0 & \text{if } n = 0 \\ \overline{d}_n - \frac{1}{n}a(0,n)\overline{c} & \text{if } n \neq 0 \end{cases}$$
$$c' = \overline{c}$$

Then $\{c'\} \cup \{d'_n \mid n \in \mathbb{Z}\}$ is a new basis for $\widetilde{\mathfrak{d}}$. The new commutation relations are

$$\begin{aligned} [c', d'_n] &= 0 \\ [d'_m, d'_n] &= (m-n)d_{m+n} + a(m,n)c = \\ &= (m-n)d'_{m+n} + a'(m,n)c' \end{aligned} \tag{12}$$

for $m, n \in \mathbb{Z}$, where $a' : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is defined by

$$a'(m,n) = \begin{cases} a(m,n) & \text{if } m+n = 0 \\ a(m,n) + \frac{m-n}{m+n}a(0,m+n) & \text{if } m+n \neq 0 \end{cases} \tag{13}$$

Note that since a is antisymmetric, so is a', and therefore in particular $a'(0,0) = 0$. From (13) follows that $a'(0,n) = 0$ for any nonzero n. These facts together with (12) shows that

$$[d'_0, d'_n] = -nd'_n \tag{14}$$

Using now the Jacobi identity which holds in $\widetilde{\mathfrak{d}}$ we obtain

$$[[d'_0, d'_n], d'_m] + [[d'_n, d'_m], d'_0] + [[d'_m, d'_0], d'_n] = 0$$

$$[-nd'_n, d'_m] + [(n-m)d'_{n+m} + a'(n,m)c', d'_0] - [d'_n, md'_m] = 0$$

$$-(n+m)(n-m)d'_{n+m} - (n+m)a'(n,m)c' + (n-m)(n+m)d'_{n+m} = 0$$

which shows that $a'(n,m) = 0$ unless $n + m = 0$. Thus, setting $b(m) = a'(m,-m)$, equation (12) can be written

$$\begin{aligned} [c', d'_n] &= 0 \\ [d'_m, d'_n] &= (m-n)d'_{m+n} + \delta_{m+n,0}b(m)c' \end{aligned}$$

Again we use Jacobi identity

$$[[d'_n, d'_1], d'_{-n-1}] + [[d'_1, d'_{-n-1}], d'_n] + [[d'_{-n-1}, d'_n], d'_1] = 0$$

$$[(n-1)d'_{n+1}, d'_{-n-1}] + [(n+2)d'_{-n}, d'_n] + [(-2n-1)d'_{-1}, d'_1] = 0$$

$$(n-1)(2(n+1)d'_0 + b(n+1)c') + (n+2)(-2nd'_0 + b(-n)c') + (-2n-1)(-2d'_0 + b(-1)c') = 0$$

$$(2n^2 - 2 - 2n^2 - 4n + 4n + 2)d'_0 + \big((n-1)b(n+1) - (n+2)b(n) + (2n+1)b(1)\big)c' = 0,$$

which is equivalent to

$$(n-1)b(n+1) = (n+2)b(n) - (2n+1)b(1).$$

This is a second order linear recurrence equation in b. One verifies that $b(m) = m$ and $b(m) = m^3$ are two solutions, obviously linear independent. Thus there are $\alpha, \beta \in \mathbb{C}$ such that

$$b(m) = \alpha m^3 + \beta m.$$

Finally, we set

$$d_n = d'_n + \delta_{n,0} \frac{\alpha + \beta}{2} c',$$

and

$$c = 12\alpha c'.$$

If $\alpha \neq 0$, this is again a change of basis. Then, for $m + n \neq 0$,

$$\begin{aligned} [d_m, d_n] &= (m-n)d'_{m+n} + \delta_{m+n,0}(\alpha m^3 + \beta m)c' = \\ &= (m-n)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c, \end{aligned}$$

and for $m + n = 0$,

$$\begin{aligned} [d_m, d_n] &= (m-n)d'_{m+n} + (\alpha m^3 + \beta m)c' = \\ &= 2md'_{m+n} + 2m \frac{\alpha + \beta}{2} c' + (\alpha m^3 - \alpha m)c' = \\ &= 2md_{m+n} + \frac{m^3 - m}{12} c = \\ &= (m-n)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c. \end{aligned}$$

From these calculations we also see that $\alpha = 0$ corresponds to the trivial extension. The proof of uniqueness is finished. To prove existence, it is enough to check that the relations (9)-(10) define a Lie algebra, which is easy. □

The antilinear map $\omega : \mathrm{Vir} \to \mathrm{Vir}$ defined by requiring

$$\omega(d_n) = d_{-n} \tag{15}$$

$$\omega(c) = c \tag{16}$$

is an antilinear anti-involution on Vir. Indeed

$$\begin{aligned} [\omega(d_n), \omega(d_m)] = [d_{-n}, d_{-m}] &= (-n+m)d_{-n-m} + \delta_{-n,m} \frac{-n^3 + n}{12} c = \\ &= (m-n)d_{-(m+n)} + \delta_{m,-n} \frac{m^3 - m}{12} c = \omega([d_m, d_n]). \end{aligned}$$

Contravariance of Hermitian forms on representations of Vir, and unitarity of the representations will always be considered with respect to this ω.

Note that Vir has the following triangular decomposition into Lie subalgebras:

$$n^- = \bigoplus_{i=1}^{\infty} \mathbb{C}d_{-i} \qquad \mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}d_0 \qquad n^+ = \bigoplus_{i=1}^{\infty} \mathbb{C}d_i \tag{17}$$

3 Representations of Vir

3.1 Positive-energy and highest weight representations

Definition 4 (Positive-energy representation of Vir)**.** Let $\pi : \mathrm{Vir} \to \mathfrak{gl}(V)$ be a representation of Vir in a linear space V such that

a) V admits a basis consisting of eigenvectors of $\pi(d_0)$,

b) all eigenvalues of the basis vectors are non-negative, and

c) the eigenspaces of $\pi(d_0)$ are finite-dimensional.

Then π is said to be a *positive-energy* representation of Vir.

Definition 5 (Highest weight representation of Vir)**.** A representation of Vir in a linear space V is a *highest weight* representation if there is an element $v \in V$ and two numbers $C, h \in \mathbb{C}$, such that

$$cv = Cv, \tag{18}$$

$$d_0 v = hv, \tag{19}$$

$$V = \mathcal{U}(\mathrm{Vir})v = \mathcal{U}(n^-)v, \tag{20}$$

$$n^+ v = 0. \tag{21}$$

The vector v is called a *highest weight vector* and (C, h) is the *highest weight*.

Remark 3. The second equality in condition (20) follows from (18), (19) and (21). To see this, use the Poincaré-Birkhoff-Witt theorem:

$$\mathcal{U}(\mathrm{Vir}) = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(n^+),$$

and write $\mathcal{U}(n^+) = \mathbb{C} \cdot 1 + \mathcal{U}(n^+)n^+$. Then

$$\mathcal{U}(\mathrm{Vir})v = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})(\mathbb{C} \cdot 1 + \mathcal{U}(n^+)n^+)v = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})v = \mathcal{U}(n^-)v,$$

where we used (21) in the second equality, and (18)-(19) in the last.

Proposition 4. *Any highest weight representation V with highest weight (C,h) has the decomposition*

$$V = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} V_{h+k} \tag{22}$$

where V_{h+k} is the $(h+k)$-eigenspace of d_0 spanned by vectors of the form

$$d_{-i_s}\dots d_{-i_1}(v) \quad \textit{with} \quad 0 < i_1 \leq \dots \leq i_s, \quad i_1+\dots+i_s = k.$$

Proof. Using that $[d_0,\cdot]$ is a derivation of $\mathcal{U}(\mathrm{Vir})$ we get

$$\begin{aligned} d_0 d_{-i_s}\dots d_{-i_1} - d_{-i_s}\dots d_{-i_1} d_0 &= \sum_{m=1}^{s} d_{-i_s}\dots d_{-i_{m+1}}[d_0, d_{-i_m}]d_{-i_{m-1}}\dots d_{-i_1} = \\ &= \sum_{m=1}^{s} i_m d_{-i_s}\dots d_{-i_{m+1}} d_{-i_m} d_{-i_{m-1}}\dots d_{-i_1} = \\ &= (i_1+\dots+i_s) d_{-i_s}\dots d_{-i_1}. \end{aligned} \tag{23}$$

Therefore we have

$$\begin{aligned} d_0(d_{-i_s}\dots d_{-i_1}(v)) &= (i_1+\dots+i_s)d_{-i_s}\dots d_{-i_1}(v) + d_{-i_s}\dots d_{-i_1}d_0(v) = \\ &= (i_1+\dots+i_s+h)d_{-i_s}\dots d_{-i_1}(v). \end{aligned}$$

□

Proposition 5. *An irreducible positive energy representation of* Vir *is a highest weight representation.*

Proof. Let $\mathrm{Vir} \to \mathfrak{gl}(V)$ be an irreducible positive energy representation of Vir in a linear space V, and let $w \in V$ be a nontrivial eigenvector for d_0. Then $d_0 w = \lambda w$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Now for any $t \in \mathbb{Z}_{\geq 0}$ and $(j_t,\dots,j_1) \in \mathbb{Z}^t$ we have, using the same calculation as in Proposition 4,

$$d_0 d_{j_t}\dots d_{j_1} w = (\lambda - (j_t+\dots+j_1)) d_{j_t}\dots d_{j_1} w.$$

Since V is a positive energy representation, this shows that the set

$$M = \{ j \in \mathbb{Z} \mid d_{j_t}\dots d_{j_1} w \neq 0 \text{ for some } t \geq 0, (j_t,\dots,j_1) \in \mathbb{Z}^t \text{ with } j_t+\dots+j_1 = j \}$$

is bounded from above by λ. It is also nonempty, because $0 \in M$. Let $t \geq 0$ and $(j_t,\dots,j_1) \in \mathbb{Z}^t$ with $j_t+\dots+j_1 = \max M$ be such that $v = d_{j_t}\dots d_{j_1} w \neq 0$. Then

$$d_j v = d_j d_{j_t}\dots d_{j_1} w = 0 \quad \text{for } j > 0 \tag{24}$$

since otherwise $j + \max M = j + j_t + \ldots + j_1 \in M$, which is impossible. We also have

$$d_0 v = d_0 d_{j_t} \ldots d_{j_1} w = (\lambda - (j_t + \ldots + j_1)) d_{j_t} \ldots d_{j_1} w = hv \tag{25}$$

where we set $h = \lambda - (j_t + \ldots + j_1)$. Using some argument involving restrictions to eigenspaces, it can be shown using Schur's Lemma that c acts by some multiple $C \in \mathbb{C}$ of the identity operator on V. In particular we have

$$cv = Cv. \tag{26}$$

Consider the submodule V' of V defined by

$$V' = \mathcal{U}(\mathrm{Vir})v. \tag{27}$$

It is nontrivial, since $0 \neq v \in V'$. Therefore, since V is irreducible, we must have $V = V'$. Recalling Remark 3 and using (24)-(27), it now follows that V is a highest weight representation, and the proof is finished. □

Proposition 6. *A unitary highest weight representation V of* Vir *is irreducible.*

Proof. If U is a subrepresentation of V, then $V = U \oplus U^\perp$. Using the decomposition (22) of V and Lemma 1 we obtain

$$U = \bigoplus_{k \geq 0} U \cap V_{h+k} \qquad\qquad U^\perp = \bigoplus_{k \geq 0} U^\perp \cap V_{h+k}$$

In particular, since V_h is one-dimensional and spanned by some nonzero highest weight vector v, we have either $v \in U$ or $v \in U^\perp$. Thus either $U = V$ or $U = 0$. □

3.2 Verma representations

Definition 6 (Verma representation of Vir)**.** A highest weight representation $M(C, h)$ of Vir with highest weight vector v and highest weight (C, h) is called a *Verma representation* if it satisfies the following universal property:

For any highest weight representation V of Vir with heighest weight vector u and highest weight (C, h), there exists a unique epimorphism $\varphi : M(C, h) \to V$ of Vir-modules which maps v to u.

Proposition 7. *For each $C, h \in \mathbb{C}$ there exists a unique Verma representation $M(C, h)$ of* Vir *with highest weight (C, h). Furthermore, the map $\mathcal{U}(n^-) \to M(C, h)$ sending x to xv is not only surjective, but also injective.*

Proof. To prove existence, let $I(C,h)$ denote the left ideal in $\mathcal{U}(\mathrm{Vir})$ generated by the elements $\{d_n \mid n > 0\} \cup \{d_0 - h \cdot 1_{\mathcal{U}(\mathrm{Vir})},\ c - C \cdot 1_{\mathcal{U}(\mathrm{Vir})}\}$, where $1_{\mathcal{U}(\mathrm{Vir})}$ is the identity element in $\mathcal{U}(\mathrm{Vir})$. Form the linear space $M(C,h) = \mathcal{U}(\mathrm{Vir})/I(C,h)$, and define a map $\pi : \mathrm{Vir} \to \mathfrak{gl}(M(C,h))$ by

$$\pi(x)(u + I(C,h)) = xu + I(C,h).$$

Then π is a representation of Vir. Furthermore, it is a highest weight representation of Vir with highest weight vector $v = 1_{\mathcal{U}(\mathrm{Vir})} + I(C,h)$ and highest weight (C,h).

We now show that π is a Verma representation. Let $\rho : \mathrm{Vir} \to \mathfrak{gl}(V)$ be any highest weight representation with highest weight (C,h) and highest weight vector u. By restricting the multiplication we can view $\mathcal{U}(\mathrm{Vir})$ as a left Vir-module. The action of $\mathcal{U}(\mathrm{Vir})$ on V

$$\begin{aligned} \alpha : \mathcal{U}(\mathrm{Vir}) &\to V \\ x &\to xu \end{aligned}$$

then becomes a Vir-module homomorphism. We claim that $\alpha(I(C,h)) = 0$. Indeed, it is enough to check that the image under α of the generators $d_n, n > 0$, $d_0 - h \cdot 1_{\mathcal{U}(\mathrm{Vir})}$, and $c - C \cdot 1_{\mathcal{U}(\mathrm{Vir})}$ of the left ideal are zero, and this follows since V is a highest weight representation of Vir with highest weight vector u and highest weight (C,h). Thus α induces a Vir-module epimorphism $\varphi : \mathcal{U}(\mathrm{Vir})/I(C,h) = M(C,h) \to V$ which clearly maps v to u. This shows existence of the map φ.

Next we prove that there can exist at most one Vir-module epimorphism $\varphi : M(C,h) \to V$ which maps v to u. Since $M(C,h)$ is a highest weight module, any element is a linear combination of elements of the form

$$d_{-i_s} \ldots d_{-i_1} + I(C,h),$$

where $i_j > 0$ and $s \geq 0$. We show by induction on s that φ is uniquely defined on each such element. If $s = 0$, we must have $\varphi(1_{\mathcal{U}(\mathrm{Vir})} + I(C,h)) = \varphi(v) = u$. If $s > 0$ we have

$$\begin{aligned} \varphi(d_{-i_s} \ldots d_{-i_1} + I(C,h)) &= \varphi(\pi(d_{-i_s})(d_{-i_{s-1}} \ldots d_{-i_1} + I(C,h) = \\ &= \rho(d_{-i_s})\varphi(d_{-i_{s-1}} \ldots d_{-i_1} + I(C,h)) \end{aligned}$$

since φ is a Vir-module homomorphism. By induction on s, φ is uniquely defined on $M(C,h)$. Consequently, π is a Verma representation.

Uniqueness of the Verma representaion $M(C,h)$ is a standard exercise in abstract nonsense. Injectivity of the map $\mathcal{U}(n^-) \ni x \mapsto \pi(x)(1_{\mathcal{U}(\mathrm{Vir})} + I(C,h)) = x + I(C,h)$ follows from the Poincaré-Birkhoff-Witt theorem. $\square$

In the rest of the article, v shall always denote a fixed choice of a nonzero highest weight vector in $M(C,h)$.

Proposition 8. *a) The Verma representation $M(C,h)$ has the decomposition*

$$M(C,h) = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} M(C,h)_{h+k} \tag{28}$$

where $M(C,h)_{h+k}$ is the $(h+k)$-eigenspace of d_0 of dimension $p(k)$ spanned by vectors of the form

$$d_{-i_s}\dots d_{-i_1}(v) \quad \textit{with} \quad 0 < i_1 \leq \dots \leq i_s, i_1 + \dots + i_s = k$$

b) M(C,h) is indecomposable, i.e. we cannot find nontrivial subrepresentations W_1, W_2 of $M(C,h)$ such that

$$M(C,h) = W_1 \oplus W_2.$$

c) M(C,h) has a unique maximal proper subrepresentation $J(C,h)$, and

$$V(C,h) = M(C,h)/J(C,h)$$

is the unique irreducible highest weight representation with highest weight (C,h).

Proof. Part (a) is a restatement of Proposition 4 for Verma modules. It remains to determine the dimension of an eigenspace V_{h+k} of d_0. Note that in a Verma representation, the set of all the vectors

$$d_{-i_s}\dots d_{-i_1}(v), \quad i_s \geq \dots \geq i_1 \geq 1, \quad i_1 + \dots + i_s = k$$

form a basis for V_{h+k} because a vanishing linear combination would contradict the injectivity of the linear map $\mathcal{U}(n^-) \ni x \mapsto xv \in M(C,h)$. The number of such vectors are precisely the number of partitions of k into positive integers.

For part b), assume that $M(C,h) = W_1 \oplus W_2$ is a decomposition into subrepresentations. Using Lemma 1 with $\mathfrak{g} = \text{Vir}$ and $\mathfrak{h} = \mathbb{C}d_0$ and $V = M(C,h)$ and $U = W_1$ and $U = W_2$, we would have

$$W_1 = \bigoplus_{k\geq 0} W_1 \cap M(C,h)_{h+k} \qquad\qquad W_2 = \bigoplus_{k\geq 0} W_2 \cap M(C,h)_{h+k}$$

respectively. Since $\dim M(C,h)_h = 1$, we have either $M(C,h)_h \subseteq W_1$ or $M(C,h)_h \subseteq W_2$. In the former case, $v \in W_1$ which imply, since W_1 is a representation of Vir, that $M(C,h) = \mathcal{U}(Vir)v \subseteq W_1$. In other words, $W_1 = M(C,h)$ and $W_2 = 0$. The other case is symmetric. Thus no nontrivial decompositions can exist.

To prove c), we observe from the proof of part b) that a subrepresentation of $M(C,h)$ is proper if and only if it does not contain the highest weight vector v. Thus if we form the sum $J(C,h)$ of all proper subrepresentations of $M(C,h)$, it is itself a proper subrepresentation of $M(C,h)$. Clearly $J(C,h)$ is maximal among all proper subrepresentations.

It is also unique, because it contains and is contained in any other maximal proper subrepresentation of $M(C,h)$.

For the uniqueness of $V(C,h)$, let $V'(C,h)$ be any irreducible highest weight module with the same highest weight (C,h). Then by definition of the Verma module there is a submodule $J'(C,h)$ of $M(C,h)$ such that

$$V'(C,h) = M(C,h)/J'(C,h).$$

Since $V'(C,h)$ is irreducible, $J'(C,h)$ must be maximal and proper, and hence equal to $J(C,h)$. Thus $V'(C,h) = V(C,h)$, and the proof is finished. □

3.3 Shapovalov's form

Proposition 9. *Let $C, h \in \mathbb{R}$. Then*

a) there is a unique contravariant Hermitian form $\langle\cdot|\cdot\rangle$ on M(C,h) such that $\langle v|v\rangle = 1$,

b) the eigenspaces of d_0 are pairwise orthogonal with respect to this form,

c) $J(C,h) = \ker\langle\cdot|\cdot\rangle \equiv \{u \in M(C,h) \mid \langle u|w\rangle = 0$ for all $w \in M(C,h)\}$.

The form is called *Shapovalov's form*.

Proof. a) We first prove uniqueness of the form. The antilinear anti-involution $\omega : \mathrm{Vir} \to \mathrm{Vir}$ defined in equations (15)-(16) extends uniquely to an antilinear anti-involution $\widetilde{\omega} : \mathcal{U}(\mathrm{Vir}) \to \mathcal{U}(\mathrm{Vir})$ on the universal enveloping algebra as follows:

$$\widetilde{\omega}(x_1 \dots x_m) = \omega(x_m) \dots \omega(x_1)$$

for elements $x_i \in \mathrm{Vir}$. If $x, y \in \mathcal{U}(\mathrm{Vir})$, then

$$\langle xv|yv\rangle = \langle v|\widetilde{\omega}(x)yv\rangle \tag{29}$$

since the form is contravariant.

The universal enveloping algebra $\mathcal{U}(\mathrm{Vir})$ of Vir has the following decomposition:

$$\mathcal{U}(\mathrm{Vir}) = (n^-\mathcal{U}(\mathrm{Vir}) + \mathcal{U}(\mathrm{Vir})n^+) \oplus \mathcal{U}(\mathfrak{h}).$$

Since $\mathfrak{h}$ is commutative, we can identify $\mathcal{U}(\mathfrak{h})$ with $S(\mathfrak{h})$, the symmetric algebra on the vectorspace $\mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}d_0$. Let $P : \mathcal{U}(\mathrm{Vir}) \to S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$ be the projection, and let $e_{(C,h)} : S(\mathfrak{h}) \to \mathbb{C}$ be the algebra homomorphism determined by

$$e_{(C,h)}(c) = C \qquad\qquad e_{(C,h)}(d_0) = h$$

Then we have for $x \in \mathcal{U}(\mathrm{Vir})$,

$$P(x)v = e_{(C,h)}(P(x))v$$

Since $M(C,h)$ is a highest weight representation, we have

$$\langle v|n^-\mathcal{U}(\mathrm{Vir})v + \mathcal{U}(\mathrm{Vir})n^+v\rangle = \langle n^+v|\mathcal{U}(\mathrm{Vir})v\rangle + \langle v|\mathcal{U}(\mathrm{Vir})n^+v\rangle = 0$$

Therefore

$$\langle xv|yv\rangle = \langle v|\widetilde{\omega}(x)yv\rangle = \langle v|P(\widetilde{\omega}(x)y)v\rangle = e_{(C,h)}\big(P(\widetilde{\omega}(x)y)\big). \tag{30}$$

This shows that the form is unique, if it exists.

To show existence, we recall the construction of $M(C,h)$ as a quotient of $\mathcal{U}(\mathrm{Vir})$ by a left ideal $I(C,h)$. Clearly $P(n^+) = P(n^-) = 0$, but we also have

$$e_{(C,h)}(P(c - C\cdot 1)) = e_{(C,h)}(c - C\cdot 1) = C - C = 0$$

$$e_{(C,h)}(P(d_0 - h\cdot 1)) = e_{(C,h)}(d_0 - h\cdot 1) = h - h = 0$$

where $1 = 1_{\mathcal{U}(\mathrm{Vir})}$. Note further that

$$P(xy) = P(x)y \qquad P(yx) = yP(x)$$

for $x \in \mathcal{U}(\mathrm{Vir})$, $y \in \mathcal{U}(\mathfrak{h})$. Combining these observations we deduce

$$e_{(C,h)}(P(x)) = 0 \qquad \text{for} \quad x \in I(C,h) \text{ or } x \in \widetilde{\omega}(I(C,h)). \tag{31}$$

It is now clear that we may take (30) as the definition of the form, because if $xv = x'v$ and $yv = y'v$ for some $x, x', y, y' \in \mathcal{U}(\mathrm{Vir})$ then $x - x', y - y' \in I(C,h)$ so that

$$\begin{aligned}\langle xv|yv\rangle - \langle x'v|y'v\rangle &= \langle (x-x')v|yv\rangle + \langle x'v|(y-y')v\rangle = \\ &= \langle \widetilde{\omega}(y)(x-x')v|v\rangle + \langle v|\widetilde{\omega}(x')(y-y')v\rangle = \\ &= 0.\end{aligned}$$

It is easy to see that the form is Hermitian. Contravariance is also clear:

$$\langle xyv|zv\rangle = e_{(C,h)}\big(P(\widetilde{\omega}(xy)z)\big) = e_{(C,h)}\big(P(\widetilde{\omega}(y)\widetilde{\omega}(x)z)\big) = \langle yv|\widetilde{\omega}(x)zv\rangle.$$

Finally, we have

$$\langle v|v\rangle = e_{(C,h)}(P(1\cdot 1)) = 1,$$

which concludes the proof of part a).

b) If $x \in M(C,h)_{h+k}$ and $y \in M(C,h)_{h+l}$ with $k \neq l$ we have

$$(k-l)\langle x|y\rangle = \langle (h+k)x|y\rangle - \langle x|(h+l)y\rangle = \langle d_0x|y\rangle - \langle x|d_0y\rangle = \langle x|\omega(d_0)y - d_0y\rangle = 0$$

since $\omega(d_0) = d_0$, and therefore we must have $\langle x|y\rangle = 0$.

c) It is easy to see, using contravariance of the form, that $\ker\langle\cdot|\cdot\rangle$ is a Vir subrepresentation of $M(C,h)$. Since $\langle v|v\rangle = 1$, it is a proper subrepresentation. Hence $\ker\langle\cdot|\cdot\rangle \subseteq J(C,h)$.

Conversely, suppose $x \in \mathcal{U}(\mathrm{Vir})$ is such that $xv \in J(C,h)$, but $xv \notin \ker\langle\cdot|\cdot\rangle$. Then there is a $y \in \mathcal{U}(\mathrm{Vir})$ such that

$$0 \neq \langle yv|xv\rangle = e_{(C,h)}(P(\widetilde{\omega}(y)x)).$$

Since $J(C,h)$ is a representation of Vir, we have found $z = \widetilde{\omega}(y)xv \in J(C,h)$ with a nonzero component in $M(C,h)_h = \mathbb{C}v$. Therefore, using Lemma 1, we must have $v \in J(C,h)$. This contradicts $J(C,h) \neq M(C,h)$ and the proof is finished. □

Corollary 10. *If $C, h \in \mathbb{R}$, then V(C,h)=M(C,h)/J(C,h) carries a unique contravariant Hermitian form $\langle\cdot|\cdot\rangle$ such that $\langle v + J(C,h)|v + J(C,h)\rangle = 1$.*

From now on we will always assume that $C, h \in \mathbb{R}$ so that the Shapovalov form is always defined.

4 Unitarity and degeneracy of representations

The unique irreducible highest weight representation $V(C,h)$ with highest weight (C,h) is called a *degenerate representation* if $V(C,h) \neq M(C,h)$. In this section we will investigate for which highest weights (C,h) the representation $V(C,h)$ is degenerate.

We will also study unitary highest weight representations. From the preceeding section we can draw some simple but important conclusions.

Proposition 11. *There exists at most one unitary highest weight representation of* Vir *for a given highest weight (C,h), namely $V(C,h)$.*

Proof. Use Proposition 6, and Proposition 8 part c). □

Thus to study unitary highest weight representations, it is enough to consider those of the irreducible representations $V(C,h)$ which are unitary. This leads to the question: For which highest weights (C,h) is $V(C,h)$ unitary? We have the following preliminary result.

Proposition 12. *If $V(C,h)$ is unitary, then $C \geq 0$ and $h \geq 0$.*

Proof. A necessary condition for unitarity of $V(C,h)$ is that

$$c_n = \langle d_{-n}v|d_{-n}v\rangle \geq 0 \qquad \text{for } n > 0.$$

Since the form is contravariant we have

$$c_n = \langle v|d_n d_{-n} v\rangle = \langle v|\big(d_{-n}d_n + 2nd_0 + \frac{n^3-n}{12}c\big)v\rangle = 2nh + \frac{n^3-n}{12}C$$

Since $c_1 = 2h$, we must have $h \geq 0$. Also, if n is sufficently large, c_n has the same sign as C, so $C \geq 0$ is also necessary. □

To give a more detailed answer, we consider the matrix $S(C,h)$ of the Shapovalov form on $M(C,h)$.

$$S(C,h) = \big(\langle d_{-i_s}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_1}v\rangle\big)_{1\leq i_1\leq\ldots\leq i_s,\ 1\leq j_1\leq\ldots\leq j_t}$$

Since $M(C,h)$ is a direct sum of finite-dimensional pairwise orthogonal subspaces $M(C,h)_{h+n}$, $n \geq 0$, the matrix $S(C,h)$ is also a direct sum of matrices $S_n(C,h)$, $n \geq 0$, where $S_n(C,h)$ is the matrix of the Shapovalov form restricted to $M(C,h)_{h+n}$.

$$S_n(C,h) = \big(\langle d_{-i_s}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_1}v\rangle\big)_{(i_1,\ldots,i_s),(j_1,\ldots,j_t)\in P(n)}, \tag{32}$$

where $P(n)$ denotes the set of all partitions of n. We now define

$$\det{}_n(C,h) = \det S_n(C,h) \tag{33}$$

A necessary and sufficient condition for the degeneracy of $V(C,h)$ is that $J(C,h) \neq 0$, and this happens if and only if $\det_n(C,h) = 0$ for some $n \geq 0$. If $V(C,h)$ is unitary, $S_n(C,h)$ must be positive semi-definite for each $n \geq 0$, and thus $\det_n(C,h)$ must be non-negative for $n \geq 0$.

The following proposition shows that the representation theory for Vir is more interesting than that of the Witt algebra.

Proposition 13 (Gomes)**.** *If $C = 0$, the only unitary highest weight representation π with heighest weight (C,h) is the trivial one which satisfies $\pi(d_n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Suppose $V(0,h)$ is unitary, and let $N \in \mathbb{Z}_{\geq 0}$. Then it is necessary that $S_{2N}(0,h)$ is positive semi-definite. In particular the matrix

$$\begin{bmatrix} \langle d_{-2N}v|d_{-2N}v\rangle & \langle d_{-N}^2v|d_{-2N}v\rangle \\ \langle d_{-2N}v|d_{-N}^2v\rangle & \langle d_{-N}^2v|d_{-N}^2v\rangle \end{bmatrix} \tag{34}$$

must be positive semi-definite. Since $C = 0$ we have

$$\langle d_{-2N}v|d_{-2N}v\rangle = \langle v|(4Nd_0 + \frac{(2N)^3-2N}{12}c)v\rangle = 4Nh,$$

$$\begin{aligned}\langle d_{-N}^2 v | d_{-2N} v\rangle = \langle d_{-2N} v | d_{-N}^2 v\rangle &= \langle v | d_{2N} d_{-N}^2 v\rangle = \\ &= \langle v | (3N d_N d_{-N} + d_{-N} 3N d_N) v\rangle = \\ &= 3N \cdot 2Nh = \\ &= 6N^2 h,\end{aligned}$$

$$\begin{aligned}\langle d_{-N}^2 v | d_{-N}^2 v\rangle &= \langle d_{-N} v | (2N d_0 d_{-N} + d_{-N} 2N d_0) v\rangle = \\ &= 2N(h + N + h)\langle d_{-N} v | d_{-N} v\rangle = \\ &= (4Nh + 2N^2) \cdot 2Nh = \\ &= 8N^2 h^2 + 4N^3 h.\end{aligned}$$

Consequently the matrix (34) has the determinant

$$(4Nh)(8N^2h^2 + 4N^3) - (6N^2h)^2 = 32N^3h^3 + 16N^4h^2 - 36N^4h^2 = 4N^3h^2(8h - 5N),$$

which is negative for sufficently large N, unless $h = 0$. By uniqueness, $V(0,0)$ must be the trivial one-dimensional representation. $\square$

Our next goal is to find a general formula for $\det_n(C,h)$. For this we will need a series of lemmas.

4.1 Some lemmas

The universal enveloping algebra $\mathcal{U}(n^-)$ of n^- has a natural filtration

$$\mathcal{U}(n^-) = \bigcup_{k=0}^{\infty} \mathcal{U}(n^-)_{(k)} \tag{35}$$

$$\mathcal{U}(n^-)_{(0)} \subseteq \mathcal{U}(n^-)_{(1)} \subseteq \dots \tag{36}$$

$$\mathcal{U}(n^-)_{(k)} \mathcal{U}(n^-)_{(l)} \subseteq \mathcal{U}(n^-)_{(k+l)} \quad \text{for } k, l \in \mathbb{Z}_{\geq 0} \tag{37}$$

where

$$\mathcal{U}(n^-)_{(k)} = \sum_{0 \leq r \leq k} (n^-)^r = \sum_{\substack{0 \leq r \leq k \\ j_r \geq \dots j_1 \geq 1}} \mathbb{C} d_{-j_r} \dots d_{-j_1}. \tag{38}$$

For simplicity we will also use the notation

$$K_{(s)} = \mathcal{U}(\mathrm{Vir})n^+ + \mathcal{U}(n^-)_{(s-1)} d_0 + \mathcal{U}(n^-)_{(s-1)} c + \mathcal{U}(n^-)_{(s)} \qquad \text{for } s \geq 1,$$

and we note that

$$\mathcal{U}(n^-)_{(t)} K_{(s)} \subseteq K_{(t+s)} \qquad \text{for } t \geq 0, s \geq 1, \tag{39}$$

$$K_{(s)} \subseteq K_{(s+1)} \qquad \text{for } s \geq 1. \tag{40}$$

Lemma 14. *Let $i \geq 1$ and $j_s, \ldots, j_1 \geq 1$ be integers, where $s \geq 1$. Then*

$$d_i d_{-j_s} \ldots d_{-j_1} \in K_{(s)}. \tag{41}$$

Furthermore, if $i \notin \{j_1, \ldots, j_s\}$, then (41) can be replaced by the stronger conclusion

$$d_i d_{-j_s} \ldots d_{-j_1} \in \mathcal{U}(\mathrm{Vir})n^+ + \mathcal{U}(n^-)_{(s-2)}d_0 + \mathcal{U}(n^-)_{(s-2)}c + \mathcal{U}(n^-)_{(s)}, \tag{42}$$

where $\mathcal{U}(n^-)_{(-1)}$ is to be interpreted as zero.

Proof. We mainly consider (41), the case (42) being analogous. We use induction on s. If $s = 1$, we have

$$d_i d_{-j_1} = d_{-j_1} d_i + (i + j_1) d_{i-j_1} + \delta_{i,-j_1} \frac{i^3 - i}{12} c.$$

Now $d_{-j_1} d_i \in \mathcal{U}(\mathrm{Vir})n^+$ and $\delta_{i,-j_1} = 0$ since $i, j_1 \geq 1$. For the middle term $(i + j_1) d_{i-j_1}$ there are three cases. First, if $i < j_1$, then $(i+j_1)d_{i-j_1} \in \mathcal{U}(n^-)_{(1)} = \mathcal{U}(n^-)_{(s)}$. Secondly, if $i > j_1$, then $(i+j_1)d_{i-j_1} \in \mathcal{U}(\mathrm{Vir})n^+$. Finally, if $i = j_1$ (this case does not occur when proving (42)), then $(i + j_1)d_{i-j_1} = (i + j_1)d_0 \in \mathcal{U}(n^-)_{(0)}d_0 = \mathcal{U}(n^-)_{(s-1)}d_0$.

For the induction step, first note that

$$d_i d_{-j_s} \ldots d_{-j_1} = d_{-j_s} d_i d_{-j_{s-1}} \ldots d_{-j_1} + [d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1}.$$

Using the induction hypothesis and (39) we have

$$d_{-j_s} d_i d_{-j_{s-1}} \ldots d_{-j_1} \in \mathcal{U}(n^-)_{(1)} K_{(s-1)} \subseteq K_{(s)}.$$

Therefore it is enough to show that

$$[d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1} \in K_{(s)}. \tag{43}$$

This is clear if $i - j_s < 0$, since $\mathcal{U}(n^-)_s \subseteq K_{(s)}$. But (43) is also true if $i - j_s > 0$, using the induction hypothesis and (40). It remains to consider the case $i = j_s$ (this case does not occur when proving (42)). Since $[d_i, d_{-i}] = 2id_0 + \frac{i^3-i}{12}c$, we get

$$\begin{aligned}
[d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1} &= (2id_0 + \frac{i^3 - i}{12} c) d_{-j_{s-1}} \ldots d_{-j_1} = \\
&= \frac{i^3 - i}{12} d_{-j_{s-1}} \ldots d_{-j_1} c + 2i d_{-j_{s-1}} \ldots d_{-j_1} d_0 \\
&\quad + 2i(j_{s-1} + \ldots + j_1) d_{-j_{s-1}} \ldots d_{-j_1}.
\end{aligned}$$

Each of these terms belongs to the desired linear space $K_{(s)}$. □

In the next lemmas, $\langle\cdot|\cdot\rangle$ will denote the Shapovalov form on $M(C,h)$. We will fix $C \in \mathbb{R}$, and consider an expression of the form

$$\langle d_{-i_s} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v\rangle$$

as a polynomial in h. We will use the notation $\deg_h p$ for the degree of p as a polynomial in h.

Lemma 15. *Suppose we have some integers $s, t \geq 1$ and $i_{t-1}, \dots, i_1 \geq 1$. If $x \in K_{(s)}$, then*

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | xv\rangle \leq \min\{t, s\}. \tag{44}$$

Proof. To show (44), we use induction on $t+s$. If $t+s=2$, then $t=s=1$ and we have

$$xv = \alpha d_0 v + \beta cv + (\gamma d_{-k} + \delta)v = (\alpha h + \beta C + \delta)v + \gamma d_{-k} v$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $k \geq 1$. Thus

$$\langle v | xv\rangle = \alpha h + \beta C + \delta.$$

The degree of this as a polynomial in h is less than or equal to $1 = \min\{t, s\}$.

The induction step can be carried out by noting that xv is a linear combination of elements of the form

$$\begin{aligned}
w_1 &= d_{-k_{r-1}} \dots d_{-k_1} d_0 v = h d_{-k_{r-1}} \dots d_{-k_1} v,\\
w_2 &= d_{-k_{r-1}} \dots d_{-k_1} cv = C d_{-k_{r-1}} \dots d_{-k_1} v,\\
w_3 &= d_{-k_r} \dots d_{-k_1} v,
\end{aligned}$$

where $r \leq s$. By Lemma 14 we have

$$d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} \in K_{(r-1)} \subseteq K_{(s-1)}$$

$$d_{i_{t-1}} d_{-k_r} \dots d_{-k_1} \in K_{(r)} \subseteq K_{(s)}$$

and therefore,

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | w_1\rangle = \deg_h \Big(h \cdot \langle d_{-i_{t-2}} \dots d_{-i_1} v | d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} v\rangle\Big) \leq$$
$$\leq 1 + \min\{t-1, s-1\} \leq \min\{t, s\}$$

by the induction hypothesis. Similarly,

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} | w_2\rangle = \deg_h \Big(C \cdot \langle d_{-i_{t-2}} \dots d_{-i_1} v | d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} v\rangle\Big) \leq$$
$$\leq \min\{t-1, s-1\} \leq \min\{t, s\}$$

Finally,

$$\deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}|w_3\rangle = \deg_h\langle d_{-i_{t-2}}\dots d_{-i_1}v|d_{i_{t-1}}d_{-k_r}\dots d_{-k_1}v\rangle \leq \\ \leq \min\{t-1,s\} \leq \min\{t,s\}$$

This proves the induction step. □

Corollary 16. *If $i_t,\dots,i_1 \geq 1$ and $j_s,\dots,j_1 \geq 1$, where $s,t \geq 1$, then*

$$\deg_h\langle d_{-i_t}\dots d_{-i_1}v|d_{-j_s}\dots d_{-j_1}v\rangle \leq \min\{t,s\}. \tag{45}$$

Proof. Take $x = d_{i_t}d_{-j_s}\dots d_{-j_1}$ which is in $K_{(s)}$ by Lemma 14. □

We now consider the case $s = t$.

Lemma 17. *Let $t \geq 1$ be an integer.*

i) If $i_t \geq \dots \geq i_1 \geq 1$ then

$$\deg_h\langle d_{-i_t}\dots d_{-i_1}v|d_{-i_t}\dots d_{-i_1}v\rangle = t. \tag{46}$$

and the coefficient of h^t is positive.

ii) If $i_t \geq \dots \geq i_1 \geq 1$ and $j_t \geq \dots \geq j_1 \geq 1$ but $(i_t,\dots,i_1) \neq (j_t,\dots,j_1)$, then

$$\deg_h\langle d_{-i_t}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_1}v\rangle < t \tag{47}$$

Proof. We show part *i)* by induction on t. For $t = 1$ we have

$$d_{i_1}d_{-i_1}v = 2i_1d_0v + \frac{i_1^3 - i_1}{12}cv = 2i_1hv + \frac{i_1^3 - i_1}{12}Cv$$

and therefore

$$\langle d_{-i_1}v|d_{-i_1}v\rangle = \langle v|d_{i_1}d_{-i_1}v\rangle = \langle v|2i_1hv + \frac{i_1^3 - i_1}{12}Cv\rangle = 2i_1h + \frac{i_1^3 - i_1}{12}C$$

For the induction step, use the formula

$$d_{i_t}d_{-i_t}\dots d_{-i_1}v = \sum_{r=1}^{t} d_{-i_t}\dots d_{-i_{r+1}}[d_{i_t},d_{-i_r}]d_{-i_{r-1}}\dots d_{-i_1}v$$

and note that $i_t - i_r \geq 0$. We consider each term separately. If r is such that $i_r = i_t$, then

$$d_{-i_t}\dots d_{-i_{r+1}}[d_{i_t},d_{-i_r}]d_{-i_{r-1}}\dots d_{-i_1}v = \\ = d_{-i_t}\dots d_{-i_{r+1}}\big(2i_td_0 + \frac{i_t^3 - i_t}{12}c\big)d_{-i_{r-1}}\dots d_{-i_1}v = \\ = \Big(2i_t(h + i_{r-1} + \dots + i_1) + \frac{i_t^3 - i_t}{12}C\Big)d_{-i_t}\dots d_{-i_{r+1}}d_{-i_{r-1}}\dots d_{-i_1}v.$$

Thus, using the induction hypothesis,

$$\deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}v|d_{-i_t}\dots d_{-i_{r+1}}[d_{i_t},d_{-i_r}]d_{-i_{r-1}}\dots d_{-i_1}v\rangle = 1+t-1=t$$

and the coefficient of h^t is positive.

If r is such that $i_r < i_t$, then by Lemma 14 we have

$$d_{-i_t}\dots d_{-i_{r+1}}[d_{i_t},d_{-i_r}]d_{-i_{r-1}}\dots d_{-i_1} \in \mathfrak{U}(n^-)_{(t-r)}K_{(r-1)} \subseteq K_{(t-1)},$$

so it follows from Lemma 15 that

$$\deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}v|d_{-i_t}\dots d_{-i_{r+1}}[d_{i_t},d_{-i_r}]d_{-i_{r-1}}\dots d_{-i_1}v\rangle \le \min\{t,t-1\} = t-1.$$

Thus such terms do not contribute to the highest power of h.

To show (47), we use induction on t. For $t=1$ we have $i_1 \neq j_1$ so $\langle d_{-i_1}v|d_{-j_1}v\rangle = 0$, since the eigenspaces of d_0 are pairwise orthogonal. For the induction step consider the calculation

$$\begin{aligned}
&\deg_h\langle d_{-i_t}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_1}v\rangle = \\
&\qquad = \deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}v|\sum_{p=1}^{t} d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1}v\rangle \le \\
&\qquad\qquad \le \max_{1\le p\le t}\{\deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1}v\rangle\}
\end{aligned}$$

For each $p \in \{1,\dots,t\}$ we consider three cases. First, if $i_t - j_p < 0$ then

$$d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1} \in \mathfrak{U}(n^-)_{(t-p)}\mathfrak{U}(n^-)_{(1)}\mathfrak{U}(n^-)_{(p-1)} \subseteq \mathfrak{U}(n^-)_{(t)}$$

so that

$$\deg_h\langle d_{-i_{t-1}}\dots d_{-i_1}v|d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1}v\rangle \le t-1 < t \tag{48}$$

by Corollary 16. Secondly, if $i_t - j_p > 0$ then

$$d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1} \in \mathfrak{U}(n^-)_{(t-p)}K_{(p-1)} \subseteq K_{(t-1)}$$

by Lemma 14, and therefore (48) holds again, using Lemma 15. For the third case, when $i_t - j_p = 0$, we have

$$d_{-j_t}\dots d_{-j_{p+1}}[d_{i_t},d_{-j_p}]d_{-j_{p-1}}\dots d_{-j_1}v = \lambda d_{-j_t}\dots d_{-j_{p+1}}d_{-j_{p-1}}\dots d_{-j_1}v$$

where $\lambda = 2i_t(h+j_{p-1}+\dots+j_1)+\frac{i_t^3-i_t}{12}C$. We claim now that

$$(i_{t-1},\dots,i_1) \neq (j_t,\dots,j_{p+1},j_{p-1},\dots,j_1). \tag{49}$$

Assume the contrary. Then in particular $i_{t-1} = j_t$, and since $j_t \geq \ldots \geq j_1$ and $i_t \geq \ldots \geq i_1$, $i_t = j_p$ we get

$$j_t \geq \ldots \geq j_{p+1} \geq j_p = i_t \geq i_{t-1} = j_t.$$

Thus all inequalities must be equalities. Hence

$$j_{p+1} = i_{t-1} \geq \ldots \geq i_p = j_{p+1}.$$

Again all inequalities must be equalities, and consequently

$$j_k = i_l \quad \text{whenever } k, l \geq p.$$

In addition we assumed that $i_k = j_k$ for $k < p$. This contradicts $(i_t, \ldots, i_1) \neq (j_t, \ldots, j_1)$, so (49) is true. Thus we can use the induction hypothesis to conclude that

$$\begin{aligned} &\deg_h \langle d_{-i_{t-1}} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \ldots d_{-j_1} v \rangle = \\ &\qquad = 1 + \deg_h \langle d_{-i_{t-1}} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_{p+1}} d_{-j_{p-1}} \ldots d_{-j_1} v \rangle < 1 + (t-1) = t. \end{aligned}$$

The proof is finished. □

4.2 Kac determinant formula

If p and q are two complex polynomials in h, we will write

$$p \sim q$$

if their highest degree terms coincide. In other words, $p \sim q$ if and only if $\deg_h(p - q) < \min\{\deg_h p, \deg_h q\}$. It is easy to see that $\sim$ is an equivalence relation on the set of complex polynomials in h.

Proposition 18.

$$\det_n(C, h) \sim \prod_{\substack{1 \leq i_1 \leq \ldots \leq i_t \\ i_1 + \ldots + i_t = n}} \langle d_{-i_t} \ldots d_{-i_1} v | d_{-i_t} \ldots d_{-i_1} v \rangle \tag{50}$$

Proof. Let $P(n)$ denote the set of all partitions of n, and for $i \in P(n)$, let $\ell(i)$ denote the length of i. For $i = (i_1, \ldots, i_s), j = (j_1, \ldots, j_t) \in P(n)$, define

$$A_{ij} = \langle d_{-i_s} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_1} v \rangle$$

Then a standard formula for the determinant gives

$$\det_n(C, h) = \sum_{\sigma \in S_{P(n)}} (-1)^{\operatorname{sgn} \sigma} \prod_{i \in P(n)} A_{i\sigma(i)}. \tag{51}$$

We will show that the term with $\sigma = \mathrm{id}$ has strictly higher h-degree than the other terms in the sum. From Lemma 17 part i) follows that $\deg_h A_{ii} = \ell(i)$ for all $i \in P(n)$. Therefore, we have

$$\deg_h \prod_{i\in P(n)} A_{i\sigma(i)} = \sum_{i\in P(n)} \ell(i) \qquad \text{when } \sigma = \mathrm{id}\,. \tag{52}$$

It follows from Corollary 16 that

$$\deg_h A_{i\sigma(i)} \leq \min\{\ell(i), \ell(\sigma(i))\},$$

for any $\sigma \in S_{P(n)}$ and all $i \in P(n)$. Also, by trivial arithmetic,

$$\min\{\ell(i), \ell(\sigma(i))\} \leq \frac{\ell(i) + \ell(\sigma(i))}{2}, \tag{53}$$

so for any $\sigma \in S_{P(n)}$, $i \in P(n)$ it is true that

$$\deg_h A_{i\sigma(i)} \leq \frac{\ell(i) + \ell(\sigma(i))}{2}. \tag{54}$$

But when $\sigma \neq \mathrm{id}$, there is some $j \in P(n)$ such that $\sigma(j) \neq j$. If $\ell(\sigma(j)) \neq \ell(j)$, the inequality (53) is strict for $i = j$. On the other hand, if $\ell(\sigma(j)) = \ell(j)$, then we can use Lemma 17 part ii) to obtain

$$\deg_h A_{j\sigma(j)} < \ell(j) = \frac{\ell(j) + \ell(\sigma(j))}{2}$$

In either case we have

$$\deg_h A_{j\sigma(j)} < \frac{\ell(j) + \ell(\sigma(j))}{2}. \tag{55}$$

Therefore, if we sum the inequalities (54) for all partitions $i \neq j$, and add (55) to the result we get

$$\deg_h \prod_{i\in P(n)} A_{i\sigma(i)} = \sum_{i\in P(n)} \deg_h A_{i\sigma(i)} < \sum_{i\in P(n)} \frac{\ell(i) + \ell(\sigma(i))}{2} = \sum_{i\in P(n)} \ell(i), \tag{56}$$

when $\sigma \neq \mathrm{id}$. In the last equality we used that $\sigma : P(n) \to P(n)$ is a bijection. Hence, combining (52) and (56) with (51), we obtain (50), which was to be proved. □

Lemma 19. *Let $k \geq 1$ be an integer. Then*

$$[d_n, d_{-n}^k] = nkd_{-n}^{k-1}\big(n(k-1) + 2d_0 + \frac{n^2-1}{12}c\big) \tag{57}$$

for all $n \in \mathbb{Z}$.

Proof. We use induction on k. For $k = 1$, we have

$$nd_{-n}^{0}\big(n \cdot 0 + 2d_0 + \frac{n^2-1}{12}c\big) = 2nd_0 + \frac{n^3-n}{12}c = [d_n, d_{-n}]. \tag{58}$$

For the induction step, we assume that (57) holds for $k = l$. Then consider the following calculations:

$$\begin{aligned}
[d_n, d_{-n}^{l+1}] &= d_n d_{-n}^{l+1} - d_{-n}^{l+1} d_n = \\
&= \big(d_n d_{-n}^{l} - d_{-n}^{l} d_n\big) d_{-n} + d_{-n}^{l}\big(d_n d_{-n} - d_{-n} d_n\big) = \\
&= [d_n, d_{-n}^{l}] d_{-n} + d_{-n}^{l}[d_n, d_{-n}] = \\
&= nld_{-n}^{l-1}\big(n(l-1) + 2d_0 + \frac{n^2-1}{12}c\big) d_{-n} + d_{-n}^{l}(2nd_0 + \frac{n^3-n}{12}) = \\
&= nd_{-n}^{l}\big(ln(l+1) + (l+1)(2d_0 + \frac{n^2-1}{12}c)\big) = \\
&= n(l+1)d_{-n}^{l}(nl + 2d_0 + \frac{n^2-1}{12}c)
\end{aligned}$$

This shows the induction step. □

Lemma 20. *Let $k \geq 1$ be an integer. Then*

$$\langle d_{-n}^{k} v | d_{-n}^{k} v\rangle = k! n^k (2h + \frac{n^2-1}{12}C)(2h + \frac{n^2-1}{12}C + n) \cdot \ldots \cdot (2h + \frac{n^2-1}{12}C + n(k-1)) \tag{59}$$

for all $n \in \mathbb{Z}$.

Proof. We use induction on k. For $k = 1$, the right hand side of (59) equals

$$1! n^1\big(2h + \frac{n^2-1}{12}C + n(1-1)\big) = 2hn + \frac{n^3-n}{12}C$$

while the left hand side is

$$\begin{aligned}
\langle d_{-n} | d_{-n} v\rangle &= \langle v | d_n d_{-n} v\rangle = \\
&= \langle v | (d_{-n} d_n + 2nd_0 + \frac{n^3-n}{12}c) v\rangle = \\
&= \langle v | (2nh + \frac{n^3-n}{12}C) v\rangle = \\
&= 2hn + \frac{n^3-n}{12}C
\end{aligned}$$

So (59) holds for $k = 1$.

For the induction step, we suppose that (59) holds for $k = l$. Then we have

$$\begin{aligned}\langle d_{-n}^{l+1}v|d_{-n}^{l+1}v\rangle &= \langle d_{-n}^{l}v|d_n d_{-n}^{l+1}v\rangle = \\ &= \langle d_{-n}^{l}v|\Big(d_{-n}^{l+1}d_n + n(l+1)d_{-n}^{l}\big(nl + 2d_0 + \frac{n^2-1}{12}c\big)\Big)v\rangle = \\ &= n(l+1)(nl+2h+\frac{n^2-1}{12}C)\langle d_{-n}^{l}v|d_{-n}^{l}v\rangle = \\ &= n(l+1)(nl+2h+\frac{n^2-1}{12}C)\cdot l!n^l(2h+\frac{n^2-1}{12}C)(2h+\frac{n^2-1}{12}C+n)\cdot\ldots \\ &\quad \ldots\cdot(2h+\frac{n^2-1}{12}C+n(l-1)) = \\ &= (l+1)!n^{l+1}(2h+\frac{n^2-1}{12}C)(2h+\frac{n^2-1}{12}C+n)\cdot\ldots\cdot(2h+\frac{n^2-1}{12}C+nl)\end{aligned}$$

where we used Lemma 19 in the second equality. This shows the induction step and the proof is finished. □

Corollary 21.

$$\langle d_{-n}^{k}v|d_{-n}^{k}v\rangle \sim k!(2nh)^k$$

Lemma 22. *Let $i_1, \ldots, i_s, j_1, \ldots j_s \in \mathbb{Z}_{>0}$, where $i_p \neq i_q$ for $p \neq q$. Then*

$$\langle d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle \sim \langle d_{-i_s}^{j_s}v|d_{-i_s}^{j_s}v\rangle\ldots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle. \tag{60}$$

Proof. We use induction on $\sum_k j_k$. If $\sum_k j_k = 1$, then we must have $s = 1$ so (60) is trivial.

To carry out the induction step, we will use that

$$\langle d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle = \langle d_{-i_s}^{j_s-1}\ldots d_{-i_1}^{j_1}v|d_{i_s}d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle.$$

First we use the Leibniz rule to obtain

$$\begin{aligned}d_{i_s}d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v &= \Big(\sum_{p=1}^{j_s} d_{-i_s}^{j_s-p}[d_{i_s}, d_{-i_s}]d_{-i_s}^{p-1}\Big)d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v \\ &\quad + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v = \\ &= \Big(\sum_{p=1}^{j_s} 2i_s\big(h+(p-1)i_s+j_{s-1}i_{s-1}+\ldots+j_1i_1\big)+\frac{i_s^3-i_s}{12}C\Big) \\ &\quad \cdot d_{-i_s}^{j_s-1}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v = \\ &= (2i_sj_sh + A)\cdot d_{-i_s}^{j_s-1}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v,\end{aligned}$$

where A is a constant independent of h. Consequently

$$\begin{aligned}\langle d_{-i_s}^{j_s}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\dots d_{-i_1}^{j_1}v\rangle \sim 2i_sj_sh\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v\rangle \\ + \langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1}v\rangle. \qquad (61)\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}2i_sj_sh\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}{}^{j_1}v|d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v\rangle \sim \\ \sim 2i_sj_sh\langle d_{-i_s}^{j_s-1}v|d_{-i_s}^{j_s-1}v\rangle\cdot\langle d_{-i_{s-1}}^{j_{s-1}}v|d_{-i_{s-1}}^{j_{s-1}}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle \sim \\ \sim 2i_sj_sh(j_s-1)!(2i_sh)^{j_s-1}\cdot\langle d_{-i_{s-1}}^{j_{s-1}}v|d_{-i_{s-1}}^{j_{s-1}}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle \sim \\ \sim \langle d_{-i_s}^{j_s}v|d_{-i_s}^{j_s}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle. \qquad (62)\end{aligned}$$

where we used Corollary 21 two times. The result will now follow from (61)-(62) if we can show that

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1}v\rangle < j_1+\dots+j_s. \qquad (63)$$

Since $i_s \neq i_p$ for $p < s$ we have by Lemma 14 that

$$d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1} \in \mathfrak{U}(\mathrm{Vir})n^+ + \mathfrak{U}(n^-)_{(k-2)}d_0 + \mathfrak{U}(n^-)_{(k-2)}c + \mathfrak{U}(n^-)_{(k)},$$

where $k = j_1+\dots+j_s$. If $x \in \mathfrak{U}(\mathrm{Vir})n^+$, then $xv = 0$. If $x \in \mathfrak{U}(n^-)_{(k-2)}$, then

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xd_0v\rangle = 1+\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xv\rangle \leq 1+j_1+\dots+j_s-2,$$

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xcv\rangle = \deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xv\rangle \leq j_1+\dots+j_s-2,$$

by Corollary 16. Finally, if $y \in \mathfrak{U}(n^-)_{(k)}$, then

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|yv\rangle \leq j_1+\dots+j_s-1,$$

again by Corollary 16. These inequalities finishes the proof of (63) and we are done. □

Lemma 23.

$$\det_n(C,h) \sim \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\leq rs\leq n}} \langle d_{-r}^s v|d_{-r}^s v\rangle^{m(r,s)},$$

where $m(r,s)$ is the number of partitions of n in which r appears exactly s times.

Proof. Use Proposition 18 and Lemma 22. □

Proposition 24. *For fixed C, $\det_n(C,h)$ is a polynomial in h of degree*

$$\sum_{\substack{r,s\in\mathbb{Z}_{>0}\\1\le rs\le n}} p(n-rs).$$

The coefficient K of the highest power of h is given by

$$K=\prod_{\substack{r,s\in\mathbb{Z}_{>0}\\1\le rs\le n}} ((2r)^s s!)^{m(r,s)}, \tag{64}$$

and $m(r,s)$ can be calculated in terms of the partition function as follows:

$$m(r,s)=p(n-rs)-p(n-r(s+1)). \tag{65}$$

Proof. We first show (65). It is easy to see that the number of partitions of n in which r appears at least s times is $p(n-rs)$. But the number of partitions in which r appears exactly s times is equal to the number those which appears at least s times minus the number of those that appears at least $s+1$ times. Thus (65) is true.

From Lemma 23 and Corollary 21 follows that the coefficient of the highest power of h is equal to (64) and that

$$\begin{aligned}\deg_h \det_n(C,h) &= \sum_{\substack{r,s\in\mathbb{Z}_{>0}\\1\le rs\le n}} s m(r,s) =\\ &= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} s\big(p(n-rs)-p(n-r(s+1))\big) =\\ &= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} \Big(p(n-rs)+(s-1)\cdot p(n-rs)-s\cdot p(n-r(s+1))\Big) =\\ &= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} \Big(p(n-rs)-[n/r]\cdot p(n-r([n/r]+1))\Big) =\\ &= \sum_{\substack{r,s\in\mathbb{Z}_{>0}\\1\le rs\le n}} p(n-rs)\end{aligned}$$

□

Lemma 25. *Let V be a linear space of dimension n, and let $A\in \mathrm{End}(V)[t]$. Then $\det A(t)$ is divisible by t^k, where k is the dimension of $\ker A(0)$.*

Proof. Choose a basis $\{e_1, \dots e_k\}$ for the subspace $\ker A(0)$ of V and extend it to a basis $B = \{e_1, \dots e_k, e_{k+1}, \dots e_n\}$ for V. Write

$$A(t) = A_0 + A_1 t + \dots A_m t^m,$$

where $A_i \in \mathrm{End}(V)$. Let M_0 and $M(t)$ be the matrices of A_0 and $A(t)$ respectively in the basis B. Since $M_0 e_i = A(0)e_i = 0$ for $1 \le i \le k$, the first k columns of M_0 in the basis $\{e_1, \dots, e_n\}$ are zero, and therefore the first k columns of $M(t)$ are divisible by t. The result follows. □

Lemma 26. *If* $\det_n(C, h)$ *has a zero at* $h = h_0$*, then* $\det_n(C, h)$ *is divisible by*

$$(h - h_0)^{p(n-k)}$$

where k is the smallest positive integer for which $\det_k(C, h)$ *vanishes at* $h = h_0$*.*

Proof. Set $J_n(C, h) = J(C, h) \cap M(C, h)_{h+n} = \ker S_n(C, h)$. For $m \ge 1$, we have

$$\det_m(C, h_0) = 0 \iff J_m(C, h_0) \ne 0.$$

Since $\det_k(C, h_0) = 0$ we can thus pick $u \in J_k(C, h_0)$, $u \ne 0$. This u must satisfy

$$d_n u = 0 \quad \text{for } n > 0,$$

since otherwise we would have for any $w \in M(C, h_0)$,

$$\langle w | d_n u \rangle = \langle d_{-n} w | u \rangle = 0,$$

because $u \in J(C, h_0)$. But $0 \ne d_n u \in M(C, h_0)_{h_0 + k - n}$:

$$d_0 d_n u = [d_0, d_n]u + d_n d_0 u = (h_0 + k - n) d_n u$$

and this contradicts the minimality of k. Then $\mathcal{U}(\mathrm{Vir})u$ is a subrepresentation of $J(C, h_0)$. The subspace $\mathcal{U}(\mathrm{Vir})u \cap M(C, h)_{h+n}$ is spanned by the elements

$$d_{-i_s} \dots d_{-i_1} u, \qquad i_s \ge \dots i_1 \ge 1, \quad i_s + \dots + i_1 = n - k.$$

These are also linearly independent, since $\mathcal{U}(\mathrm{Vir})$ has no divizors of zero. Therefore $J_n(C, h_0)$ has a subspace of dimension $p(n-k)$, so $S_n(C, h_0)$ has a kernel of at least dimension $p(n-k)$. The result now follows from Lemma 25. □

We will need the following fact, which we will not prove.

Fact 27. $\det_n(C, h)$ *has a zero at* $h = h_{r,s}(C)$*, where*

$$h_{r,s}(C) = \frac{1}{48}\Big((13 - C)(r^2 + s^2) + \sqrt{(C-1)(C-25)}(r^2 - s^2) - 24rs - 2 + 2C\Big), \tag{66}$$

for each pair (r, s) of positive integers such that $1 \le rs \le n$.

The following is the main theorem of this article.

Theorem 28 (Kac determinant formula)**.**

$$\det_n(C,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h-h_{r,s}(C))^{p(n-rs)}, \tag{67}$$

where

$$K = \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} ((2r)^s s!)^{m(r,s)} \tag{68}$$

and

$$m(r,s) = p(n-rs) - p(n-r(s+1))$$

and $h_{r,s}$ is given by (66).

Proof. From Fact 27 follows that $\det_n(C,h)$ has a zero at $h = h_{r,s}(C)$ for each $r,s\in\mathbb{Z}_{>0}$ satisfying $1\le rs\le n$. Using Lemma 26 we deduce that $\det_n(C,h)$ is divisible by $(h-h_{r,s}(C))^{p(n-rs)}$ for each $r,s\in\mathbb{Z}_{>0}$ with $1\le rs\le n$. Hence $\det_n(C,h)$ is divisible by the right hand side of (67), as polynomials in h. But we know from Proposition 24 that the degree in h of the determinant $\det_n(C,h)$ equals the degree in h of the right hand side of (67), and that the coefficient of the highest power of h is given by (68). Therefore equality holds in (67), and the proof is finished. □

If we set

$$\varphi_{r,r}(C) = h - h_{r,r}(C), \tag{69}$$

and

$$\varphi_{r,s}(C) = (h-h_{r,s}(C))(h-h_{s,r}(C)), \tag{70}$$

for $r\neq s$, then (67) can be written

$$\det_n(C,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ s\le r\\ 1\le rs\le n}} \varphi_{r,s}(C)^{p(n-rs)}. \tag{71}$$

We will also use the following notation

$$\begin{aligned}\alpha_{r,s} &= \frac{1}{48}\big((13-C)(r^2+s^2) - 24rs - 2 + 2C\big) = \\ &= \frac{1}{4}(r-s)^2 - \frac{1}{48}(C-1)(r^2+s^2-2),\end{aligned} \tag{72}$$

$$\beta_{r,s} = \frac{1}{48}\sqrt{(C-1)(C-25)}(r^2-s^2). \tag{73}$$

Then

$$h_{r,s} = \alpha_{r,s} + \beta_{r,s}.$$

Note that α is symmetric in its indices, and β is antisymmetric. Therefore

$$\varphi_{r,s} = (h - h_{r,s})(h - h_{s,r}) = h^2 - 2\alpha_{r,s}h + \alpha_{r,s}^2 - \beta_{r,s}^2, \tag{74}$$

for $r \neq s$.

4.3 Consequences of the formula

Let us now return to the questions we asked at the beginning of Section 4.

Proposition 29. *a) $V(1,h) = M(1,h)$ if and only if $h \neq m^2/4$ for all $m \in \mathbb{Z}$.*
b) $V(0,h) = M(0,h)$ if and only if $h \neq (m^2-1)/24$ for all $m \in \mathbb{Z}$.

Proof. a) Putting $C = 1$ in (66) we get

$$h_{r,s}(1) = \frac{1}{48}\big(12(r^2+s^2) - 24rs\big) = \frac{(r-s)^2}{4}.$$

Thus, using (67) we obtain

$$\det_n(1,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h - \frac{(r-s)^2}{4})^{p(n-rs)}.$$

Therefore, $\det_n(1,h) \neq 0$ for all $n \in \mathbb{Z}$ if and only if $h \neq m^2/4$ for all $m \in \mathbb{Z}$.

b) When $C = 0$ we obtain

$$\begin{aligned} h_{r,s}(0) &= \frac{1}{48}\big(13(r^2+s^2) + 5(r^2-s^2) - 24rs - 2\big) = \\ &= \frac{9r^2+4s^2-12rs-1}{24} = \\ &= \frac{(3r-2s)^2-1}{24}. \end{aligned}$$

Hence by formula (67) we have

$$\det_n(0,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h - \frac{(3r-2s)^2-1}{24})^{p(n-rs)}.$$

This shows that $\det_n(0,h) \neq 0$ for all $n \in \mathbb{Z}$ if and only if $h \neq (m^2-1)/24$ for all $m \in \mathbb{Z}$. $\square$

We need the following fact which we will not prove.

Fact 30. *$V(1,3)$ is unitary.*

Then we have the following proposition.

Proposition 31. *a) $V(C,h) = M(C,h)$ for $C > 1,\ h > 0$.*
b) $V(C,h)$ is unitary for $C \geq 1$ and $h \geq 0$.

Proof. a) It will be enough to show that $\det_n(C,h) > 0$ for all $C > 1, h > 0$ and $n \geq 1$. We prove in fact that each factor $\varphi_{r,s}$ of the product (71) is positive. For $s = r$, $1 \leq r \leq n$ we have

$$\varphi_{r,r}(C) = h - \frac{1}{48}\big((13-c)2r^2 - 24r^2 - 2 + 2C\big) = h + \frac{1}{24}(C-1)(r^2-1) > 0,$$

if $C > 1$ and $h > 0$. For $r \neq s$ we have

$$\begin{aligned}
\varphi_{r,s} &= h^2 - 2\alpha_{r,s}h + \alpha_{r,s}^2 - \beta_{r,s}^2 = \\
&= h^2 - \frac{1}{2}(r-s)^2 h + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{16}(r-s)^4 - 2\frac{1}{4\cdot 48}(r-s)^2(C-1)(r^2+s^2-2) + \frac{1}{48^2}(C-1)^2(r^2+s^2-2)^2 \\
&\quad - \frac{1}{48^2}(C-1)(C-25)(r^2-s^2)^2 = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{48^2}(C-1)^2\big((r^2+s^2-2)^2 - (r^2-s^2)^2\big) \\
&\quad + (C-1)\big(\frac{24}{48^2}(r^2-s^2)^2 - \frac{1}{2\cdot 48}(r-s)^2(r^2+s^2-2)\big) = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{48^2}(C-1)^2\big(2r^2s^2 - 4(r^2+s^2) + 4 + 2r^2s^2\big) \\
&\quad + \frac{1}{96}(C-1)(r-s)^2\big(r^2 + 2rs + s^2 - r^2 - s^2 + 2\big) = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{12\cdot 48}(C-1)^2(r^2-1)(s^2-1) \\
&\quad + \frac{1}{48}(C-1)(r-s)^2(rs+1).
\end{aligned}$$

This expression is strictly positive when $C > 1$ and $h > 0$. Therefore, when $C > 1, h > 0$, we have $\det_n(C, h) > 0$ for all $n > 0$, which proves part a).

b) Let $C \geq 1$ and $h \geq 0$ be arbitrary. Since $\mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0}$ is pathwise connected, we can choose a path π from $(1, 3)$ to (C, h), i.e. a continuous function

$$\pi : [0, 1] \to \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0},$$

such that

$$p(0) = (1, 3) \quad \text{and} \quad p(1) = (C, h).$$

Moreover, this path can be chosen so that the image of the open interval $(0, 1)$ is contained in the open quadrant $\mathbb{R}_{>1} \times \mathbb{R}_{>0}$.

Let $n \in \mathbb{Z}_{\geq 0}$, and let

$$q(x, t) = a_n(x)t^{p(n)} + \ldots + a_0(x) = \det(S_n(\pi(x)) - tI)$$

be the characteristic polynomial of $S_n(\pi(x))$, the matrix of the Shapovalov form at level n on the Verma module with highest weight $\pi(x)$. Since $S_n(\pi(x))$ is Hermitian, each root of its characteristic equation is real. For $x \in [0, 1]$, we denote the roots by $\lambda_j(x)$, $j = 1, \ldots, p(n)$ such that

$$\lambda_1(x) \leq \ldots \leq \lambda_{p(n)}(x) \quad \text{for all } x \in [0, 1].$$

By a theorem on roots of polynomial equations, the roots are continuous functions of the coefficients. Thus, since the coefficients a_i in this case depend continuously on x, the roots $\lambda_j(x)$ of the characteristic equation of $S_n(\pi(x))$ are continuous functions of $x \in [0, 1]$. By the proof of part a) and the choice of π, we have $\det\big(S_n(\pi(x))\big) \neq 0$ for $x \in (0, 1)$. By Proposition 29 part a) we also have $\det(S_n(\pi(0))) = \det(S_n(1, 3)) \neq 0$, since $3 \neq m^2/4$ for all integers m. Thus none of the roots $\lambda_j(x)$ can be zero when $x < 1$. From Fact 30 follows that $\lambda_j(0) > 0$ for $j = 1, \ldots, p(n)$, so using the intermediate value theorem we obtain $\lambda_j(x) > 0$ for $j = 1, \ldots, p(n)$ and $x \in [0, 1)$. Hence $\lambda_j(1) \geq 0$ for $j = 1, \ldots, p(n)$. By spectral theory there is a unitary matrix U such that

$$\bar{U}^t S_n(\pi(1))U = U^{-1}S_n(\pi(1))U = \operatorname{diag}(\lambda_j(1)),$$

which shows that $S_n(\pi(1)) = S_n(C, h)$ is positive semi-definite for any $n \in \mathbb{Z}_{\geq 0}$. Thus $V(C, h)$ is unitary.

□

4.4 Calculations for $n = 3$

In this section we calculate $\det_3(C, h)$ first by hand, and then by using Kac determinant formula.

4.4.1 By hand

We have

$$\det_3(C,h) = \begin{vmatrix} \langle d_{-3}v|d_{-3}v\rangle & \langle d_{-3}v|d_{-2}d_{-1}v\rangle & \langle d_{-3}v|d_{-1}^3v\rangle \\ \langle d_{-2}d_{-1}v|d_{-3}v\rangle & \langle d_{-2}d_{-1}v|d_{-2}d_{-1}v\rangle & \langle d_{-2}d_{-1}v|d_{-1}^3v\rangle \\ \langle d_{-1}^3v|d_{-3}v\rangle & \langle d_{-1}^3v|d_{-2}d_{-1}v\rangle & \langle d_{-1}^3v|d_{-1}^3v\rangle \end{vmatrix}.$$

We calculate the entries:

$$\begin{aligned}\langle d_{-3}v|d_{-3}v\rangle &= \langle v|(6d_0 + \frac{3^3-3}{12}c)v\rangle = \\ &= 6h + 2C\end{aligned}$$

$$\begin{aligned}\langle d_{-2}d_{-1}v|d_{-3}v\rangle &= \langle d_{-1}v|5d_{-1}v\rangle = \\ &= 10h\end{aligned}$$

$$\begin{aligned}\langle d_{-1}^3v|d_{-3}v\rangle &= \langle d_{-1}^2v|4d_{-2}v\rangle = \\ &= 4\langle d_{-1}v|3d_{-1}v\rangle = \\ &= 24h\end{aligned}$$

$$\begin{aligned}\langle d_{-2}d_{-1}v|d_{-2}d_{-1}v\rangle &= \langle d_{-1}v|(4d_0 + \frac{2^3-2}{12}c)d_{-1}v + d_{-2}3d_1v\rangle = \\ &= (4(h+1) + C/2)2h = \\ &= 8h^2 + (C+8)h\end{aligned}$$

$$\begin{aligned}\langle d_{-1}^3v|d_{-2}d_{-1}v\rangle &= \langle d_{-1}^2v|3d_{-1}d_{-1}v + d_{-2}2d_0v\rangle = \\ &= 3\langle d_{-1}v|2d_0d_{-1}v + d_{-1}2d_0v\rangle + 2h\langle d_{-1}v|3d_{-1}v\rangle = \\ &= 6(h+1)2h + 6h\cdot 2h + 6h\cdot 2h = \\ &= 36h^2 + 12h\end{aligned}$$

$$\begin{aligned}\langle d_{-1}^3v|d_{-1}^3v\rangle &= \langle d_{-1}^2v|2d_0d_{-1}^2v + d_{-1}2d_0d_{-1}v + d_{-1}^22d_0v\rangle = \\ &= 2(h+2+h+1+h)\langle d_{-1}v|2d_0d_{-1}v + d_{-1}2d_0v\rangle = \\ &= 6(h+1)\cdot 2(h+1+h)\cdot 2h = \\ &= 24h(2h^2+3h+1) = \\ &= 48h^3 + 72h^2 + 24h\end{aligned}$$

Thus the determinant is equal to

$$\begin{aligned}
\det{}_3(C,h) &= \begin{vmatrix} 6h+2C & 10h & 24h \\ 10h & 8h^2+(C+8)h & 36h^2+12h \\ 24h & 36h^2+12h & 48h^3+72h^2+24h \end{vmatrix} = \\
&= 48h^2 \begin{vmatrix} 3h+C & 10h & 12h \\ 5 & 8h+C+8 & 18h+6 \\ 1 & 3h+1 & 2h^2+3h+1 \end{vmatrix} = \\
&= 48h^2\Big(12h\big(15h+5-(8h+C+8)\big) \\
&\quad -(18h+6)\big((3h+C)(3h+1)-10h\big) \\
&\quad +(2h^2+3h+1)\big((3h+C)(8h+C+8)-50h\big)\Big) = \\
&= 48h^2\Big(84h^2-(12C+36)h \\
&\quad -(18h+6)(9h^2+(3C-7)h+C) \\
&\quad +(2h^2+3h+1)(24h^2+(11C-26)h+C^2+8C)\Big) = \\
&= 48h^2\Big(84h^2-(12C+36)h \\
&\quad -\big(162h^3+(54C-72)h^2+(36C-42)h+6C\big) \\
&\quad +48h^4+(22C+20)h^3+(2C^2+49C-54)h^2 \\
&\quad\quad +(3C^2+35C-26)h+C^2+8C = \\
&= 48h^2\Big(48h^4+(22C-142)h^3+(2C^2-5C+102)h^2 \\
&\qquad +(3C^2-13C-20)h+C^2+2C\Big).
\end{aligned} \tag{75}$$

4.4.2 Using the formula

To use the determinant formula, we first calculate the coefficient K for $n = 3$. The partitions of 3 are (3), $(2,1)$ and $(1,1,1)$. Thus

$$\begin{aligned}
K &= ((2\cdot 1)^1 1!)^1 \cdot ((2\cdot 1)^2 2!)^0 \cdot ((2\cdot 2)^1 1!)^1 \cdot ((2\cdot 1)^3 3!)^1 \cdot ((2\cdot 3)^1 1!)^1 = \\
&= 2\cdot 4\cdot 8\cdot 6\cdot 6 = 48^2.
\end{aligned}$$

By (71) we now have

$$\det{}_3(C,h) = 48^2 \varphi_{1,1}^2 \varphi_{2,1}\varphi_{3,1}. \tag{76}$$

First we have

$$\varphi_{1,1}(C) = h - h_{1,1}(C) = h. \tag{77}$$

We will use the notation introduced in (73)-(72). Then

$$\alpha_{2,1} = \frac{1}{4}(2-1)^2 - \frac{3}{48}(C-1) = \frac{5}{16} - \frac{1}{16}C,$$

$$\alpha_{2,1}^2 = \frac{1}{16^2}C^2 - \frac{10}{16^2}C + \frac{25}{16^2},$$

$$\beta_{2,1}^2 = \frac{9}{48^2}(C-1)(C-25) = \frac{1}{16^2}C^2 - \frac{26}{16^2}C + \frac{25}{16^2}.$$

Hence, using (74),

$$\varphi_{2,1}(C) = h^2 + (\frac{1}{8}C - \frac{5}{8})h + \frac{1}{16}C. \tag{78}$$

Also,

$$\alpha_{3,1} = \frac{1}{4}(3-1)^2 - \frac{8}{48}(C-1) = \frac{7}{6} - \frac{1}{6}C,$$

$$\alpha_{3,1}^2 = \frac{1}{36}C^2 - \frac{14}{36}C + \frac{49}{36},$$

$$\beta_{3,1}^2 = \frac{64}{48^2}(C-1)(C-25) = \frac{1}{36}C^2 - \frac{26}{36}C + \frac{25}{36}.$$

Therefore,

$$\varphi_{3,1}(C) = h^2 + (\frac{1}{3}C - \frac{7}{3})h + \frac{1}{3}C + \frac{2}{3}. \tag{79}$$

Consequently, using (76) we have

$$\begin{aligned}
\det{}_3(C,h) &= 48^2h^2\big(h^2 + (\frac{1}{8}C - \frac{5}{8})h + \frac{1}{16}C\big)\big(h^2 + (\frac{1}{3}C - \frac{7}{3})h + \frac{1}{3}C + \frac{2}{3}\big) = \\
&= 48h^2\big(16h^2 + (2C-10)h + C\big)\big(3h^2 + (C-7)h + C + 2\big) = \\
&= 48h^2\big(48h^4 + (16C - 112 + 6C - 30)h^3 \\
&\quad + (16C + 32 + 2C^2 - 14C - 10C + 70 + 3C)h^2 \\
&\quad + (2C^2 + 4C - 10C - 20 + C^2 - 7C)h + C^2 + 2C\big) = \\
&= 48h^2\big(48h^4 + (22C - 142)h^3 + (2C^2 - 5C + 102)h^2 \\
&\quad + (3C^2 - 13C - 20)h + C^2 + 2C\big).
\end{aligned}$$

This coincides with (75).

5 The centerless Ramond algebra

Let $\mathbb{C}[x,y,z]$ be the commutative associative algebra of polynomials in three indeterminates x, y, z. Form the ideal I generated by the two elements $xy - 1$ and z^2. Let

$$A = \mathbb{C}[x,y,z]/I$$

denote the quotient algebra. We will denote the images of x, y, and z under the canonical projection $\mathbb{C}[x, y, z] \to A$ by t, t^{-1} and ϵ respectively. Then we have

$$t^{-1}t = tt^{-1} = 1 \qquad \epsilon^2 = 0.$$

The algebra A can also be thought of as the tensor product algebra of $\mathbb{C}[t, t^{-1}]$ with the exterior algebra $\Lambda(\mathbb{C}\epsilon)$ on a one-dimensional linear space.

We have a $\mathbb{Z}_2$-grading

$$A = A_0 \oplus A_1, \tag{80}$$

$$A_i A_j \subset A_{i+j}, \tag{81}$$

defined by

$$A_0 = \mathbb{C}[t, t^{-1}], \qquad A_1 = \mathbb{C}[t, t^{-1}]\epsilon.$$

Since $A_1^2 = 0$, A can also be thought of as a supercommutative algebra:

$$ab = (-1)^{|a||b|} ba \quad \text{for } a, b \in A_0 \cup A_1,$$

where $|a| \in \mathbb{Z}_2$ denotes the degree of a homogenous element $a \in A_0 \cup A_1$.

For $n \in \mathbb{Z}$ we define the linear operators L_n, F_n on A by

$$L_n = -t^{n+1}\frac{d}{dt} - \frac{n}{2}t^n\epsilon\frac{d}{d\epsilon},$$

$$F_n = it^{n+1}\epsilon\frac{d}{dt} + it^n\frac{d}{d\epsilon}.$$

More explicitly we can define these mappings by requiring

$$L_n : t^k \mapsto -kt^{n+k},$$
$$L_n : t^k\epsilon \mapsto (-k - \frac{n}{2})t^{n+k}\epsilon,$$

and

$$F_n : t^k \mapsto ikt^{n+k}\epsilon,$$
$$F_n : t^k\epsilon \mapsto it^{n+k},$$

where $i = \sqrt{-1}$.

Proposition 32. *For $n \in \mathbb{Z}$, L_n is an even superderivation on A and F_n is an odd superderivation on A, in the sence that*

$$L_n(ab) = L_n(a)b + aL_n(b)$$

$$F_n(ab) = F_n(a)b + (-1)^{|a|}aF_n(b)$$

for homogenous $a, b \in A$.

Proof. A straightforward calculation yields

$$\begin{aligned}
L_n(t^k t^l) &= L_n(t^{k+l}) = (-k-l)t^{n+k+l} = -kt^{n+k}t^l - t^k \cdot lt^{n+l} = L_n(t^k)t^l + t^k L_n(t^l),\\
L_n(t^k\epsilon t^l) &= L_n(t^{k+l}\epsilon) = (-k-l-n/2)t^{n+k+l}\epsilon = (-k-n/2)t^{n+k}\epsilon \cdot t^l - t^k\epsilon \cdot lt^{n+l} =\\
&= L_n(t^k\epsilon)t^l + t^k\epsilon L_n(t^l),\\
L_n(t^k t^l\epsilon) &= L_n(t^{k+l}\epsilon) = (-k-l-n/2)t^{n+k+l}\epsilon = -kt^{n+k} \cdot t^l\epsilon + t^k \cdot (-l-n/2)t^{n+l}\epsilon =\\
&= L_n(t^k)t^l\epsilon + t^k L_n(t^l\epsilon),\\
L_n(t^k\epsilon t^l\epsilon) &= L_n(0) = 0 = (-k-n/2)t^{n+k}\epsilon \cdot t^l\epsilon + t^k\epsilon \cdot (-l-n/2)t^{n+l}\epsilon =\\
&= L_n(t^k\epsilon)t^l\epsilon + t^k\epsilon L_n(t^l\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
F_n(t^k t^l) &= F_n(t^{k+l}) = i(k+l)t^{n+k+l}\epsilon = ikt^{n+k}\epsilon \cdot t^l + t^k \cdot ilt^{n+l}\epsilon = F_n(t^k)t^l + t^k F_n(t^l),\\
F_n(t^k\epsilon t^l) &= F_n(t^{k+l}\epsilon) = it^{n+k+l} = it^{n+k}t^l - t^k\epsilon \cdot ilt^{n+l}\epsilon = F_n(t^k\epsilon)t^l - t^k\epsilon F_n(t^l),\\
F_n(t^k t^l\epsilon) &= F_n(t^{k+l}\epsilon) = it^{n+k+l} = ikt^{n+k}\epsilon \cdot t^l\epsilon + t^k \cdot it^{n+l} = F_n(t^k)t^l\epsilon + t^k F_n(t^l\epsilon),\\
F_n(t^k\epsilon t^l\epsilon) &= F_n(0) = 0 = it^{n+k} \cdot t^l\epsilon - t^k\epsilon \cdot it^{n+l} = F_n(t^k\epsilon)t^l\epsilon - t^k\epsilon F_n(t^l\epsilon).
\end{aligned}$$

□

The *anticommutator* $[P,Q]_+$ of two linear operators P and Q on A is defined by

$$[P,Q]_+ = PQ + QP.$$

Proposition 33. *The operators L_n, F_n satisfy the following commutation relations:*

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n},\\
[L_m, F_n] &= (\frac{1}{2}m - n)F_{m+n},\\
[F_m, F_n]_+ &= 2L_{m+n}.
\end{aligned}$$

Remark 4. This shows that L_n and F_n generate a super Lie algebra. It is called the *centerless Ramond algebra*.

Proof. We have

$$\begin{aligned}
[L_m, L_n](t^k) &= (L_m L_n - L_n L_m)(t^k) =\\
&= L_m(-kt^{n+k}) - L_n(-kt^{m+k}) =\\
&= -k(-n-k)t^{m+n+k} + k(-m-k)t^{n+m+k} =\\
&= (m-n)(-k)t^{m+n+k} =\\
&= (m-n)L_{n+m}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[L_m, L_n](t^k\epsilon) &= (L_mL_n - L_nL_m)(t^k\epsilon) = \\
&= L_m((-k-n/2)t^{n+k}\epsilon) - L_n((-k-m/2)t^{m+k}\epsilon) = \\
&= (-k-n/2)(-n-k-m/2)t^{m+n+k}\epsilon \\
&\quad -(-k-m/2)(-m-k-n/2)t^{n+m+k}\epsilon = \\
&= (nk+n^2/2-mk-m^2/2)t^{m+n+k}\epsilon = \\
&= (m-n)(-k-(m+n)/2)t^{m+n+k}\epsilon = \\
&= (m-n)L_{m+n}(t^k\epsilon).
\end{aligned}$$

Also,

$$\begin{aligned}
[L_m, F_n](t^k) &= (L_mF_n - F_nL_m)(t^k) = \\
&= L_m(ikt^{n+k}\epsilon) - F_n(-kt^{m+k}) = \\
&= ik(-n-k-m/2)t^{m+n+k}\epsilon + ki(m+k)t^{n+m+k}\epsilon = \\
&= (m/2-n)ikt^{m+n+k}\epsilon = \\
&= (m/2-n)F_{m+n}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[L_m, F_n](t^k\epsilon) &= (L_mF_n - F_nL_m)(t^k\epsilon) = \\
&= L_m(it^{n+k}) - F_n((-k-m/2)t^{m+k}\epsilon) = \\
&= -i(n+k)t^{m+n+k} - (-k-m/2)it^{n+m+k} = \\
&= (m/2-n)it^{m+n+k} = \\
&= (m/2-n)F_{m+n}(t^k).
\end{aligned}$$

Finally we have,

$$\begin{aligned}
[F_m, F_n]_+(t^k) &= (F_mF_n + F_nF_m)(t^k) = \\
&= F_m(ikt^{n+k}\epsilon) + F_n(ikt^{m+k}\epsilon) = \\
&= ki^2t^{m+n+k} + ki^2t^{n+m+k} = \\
&= 2L_{m+n}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[F_m, F_n]_+(t^k\epsilon) &= (F_mF_n + F_nF_m)(t^k\epsilon) = \\
&= F_m(it^{n+k}) + F_n(it^{m+k}) = \\
&= i^2(n+k)t^{m+n+k}\epsilon + i^2(m+k)t^{n+m+k}\epsilon = \\
&= 2(-k-(m+n)/2)t^{m+n+k}\epsilon = \\
&= 2L_{m+n}(t^k).
\end{aligned}$$

The proof is finished. □

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