The Diagonal Reduction Algebra for the Lie Superalgebra $\mathfrak{osp}(1|2)$

Jonas T. Hartwig

Based on arXiv:2106.04380 and arXiv:2203.08068
Joint with Dwight Anderson Williams II

May 5, 2022

Vector Superspaces

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ group of order two

Definition

A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space

$$V = V_{\bar{0}} \oplus V_{\bar{1}}$$

Elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are *homogeneous*. For homogeneous v, the *parity* $|v| \in \mathbb{Z}_2$ is defined by

$$v \in V_{|v|}.$$

If $|v| = \bar{0}$ we say v is *even*.
If $|v| = \bar{1}$ we say v is *odd*.
(The zero vector is both even and odd.)

Associative Superalgebras

Definition

An *associative superalgebra* is a vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with a bilinear multiplication

$$A \times A \to A, \qquad (a, b) \mapsto ab$$

and an identity element $1_A \in A$ such that

$$A_i A_j \subset A_{i+j} \qquad 1_A \in A_{\bar{0}}$$

Example

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace.
Let $A = \mathrm{End}_{\mathbb{C}}(V)$ be the associative algebra of linear transformations from V to V.
Define

$$A_{\bar{0}} = \{T : V \to V \mid T(V_{\bar{0}}) \subset V_{\bar{0}}, T(V_{\bar{1}}) \subset V_{\bar{1}}\}$$
$$A_{\bar{1}} = \{T : V \to V \mid T(V_{\bar{0}}) \subset V_{\bar{1}}, T(V_{\bar{1}}) \subset V_{\bar{0}}\}$$

Then A is an associative superalgebra.

Lie Superalgebras

Definition

A *Lie Superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, y) \mapsto [x, y]$$

satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$
$$[x, y] = -(-1)^{|x||y|}[y, x]$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$$

Note

$\mathfrak{g}_{\bar{0}}$ is a Lie algebra.

The General Linear Lie Superalgebra

If A is an associative superalgebra we obtain a Lie superalgebra $\mathcal{L}(A)$ by

$$\mathcal{L}(A) = A \quad \text{as vector superspace}$$

$$[a, b] = ab - (-1)^{|a||b|} ba$$

Definition

If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace, the *general linear Lie superalgebra* is

$$\mathfrak{gl}(V) = \mathcal{L}\big(\operatorname{End}_{\mathbb{C}}(V)\big)$$

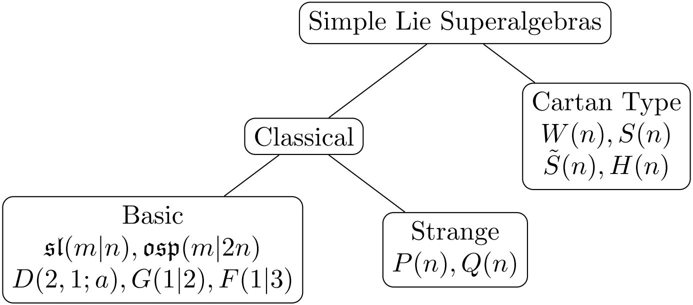
Explicitly:

$$V = \mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n$$

$$\mathfrak{gl}(V) = \mathfrak{gl}(m, n) = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\}$$

$$[x, y] = xy - (-1)^{|x||y|} yx$$

Kac's Classification



The Orthosymplectic Lie Superalgebra

V vector superspace with non-degenerate bilinear form $(\cdot,\cdot)$ which is

- *supersymmetric*: $(v,w) = (-1)^{|v||w|}(w,v)$,
- *even*: $(V_{\bar{0}}, V_{\bar{1}}) = 0$.

Definition

The *orthosymplectic Lie superalgebra* is

$$\mathfrak{osp}(V) = \{a \in \mathfrak{gl}(V) \mid (av,w) + (-1)^{|a||v|}(v,aw) = 0 \ \forall v,w \in W\}$$

When $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = 2n$ we write

$$\mathfrak{osp}(m|2n) = \mathfrak{osp}(V)$$

Note

$$\mathfrak{osp}(m|2n)_{\bar{0}} = \mathfrak{o}(m) \oplus \mathfrak{sp}(2n)$$

Representations

Definition

(i) A *representation* of a Lie superalgebra $\mathfrak{g}$ is a vector superspace V with a Lie superalgebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We write

$$x.v = \rho(x)v \qquad x \in \mathfrak{g}, v \in V.$$

(ii) A *subrepresentation* of V is a subsuperspace $U \subset V$ such that

$$x.u \in U \qquad \forall x \in \mathfrak{g}, u \in U$$

(iii) A representation V is *irreducible* if $\{0\}$ and V are the only subrepresentations.

(iv) V is *completely reducible* if V is the direct sum of irreducible subrepresentations

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$$

Tensor Products

Definition

The *tensor product* of representations V and W of $\mathfrak{g}$ is

$$V \otimes W = \left(V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}\right) \oplus \left(V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}}\right)$$

with action

$$x.(v \otimes w) = (x.v) \otimes w + (-1)^{|x||v|} v \otimes (x.w)$$

Problem

Given representations V and W of $\mathfrak{g}$, such that $V \otimes W$ is completely reducible, find an explicit decomposition

$$V \otimes W = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

into irreducible subrepresentations U_i.

The Orthosymplectic Lie Superalgebra $\mathfrak{osp}(1|2)$

$$\mathfrak{osp}(1|2) = \mathfrak{osp}(1|2)_{\bar{0}} \oplus \mathfrak{osp}(1|2)_{\bar{1}}$$

$$\mathfrak{osp}(1|2)_{\bar{0}} = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e \qquad \mathfrak{osp}(1|2)_{\bar{1}} = \mathbb{C}y \oplus \mathbb{C}x$$

$$f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{osp}(1|2) = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$$

$$\mathfrak{g}_- = \mathbb{C}f \oplus \mathbb{C}y \qquad \mathfrak{h} = \mathbb{C}h \qquad \mathfrak{g}_+ = \mathbb{C}e \oplus \mathbb{C}x$$

Irreducible Representations of $\mathfrak{osp}(1|2)$

- Every finite-dimensional representation of $\mathfrak{osp}(1|2n)$ is completely reducible.
- For each $\ell \in \mathbb{Z}_{\geq 0}$ there is an irreducible representation $V(\ell)$ of $\mathfrak{osp}(1|2)$ of dimension $2\ell + 1$.
- Every finite-dimensional irreducible representation of $\mathfrak{osp}(1|2)$ is equivalent to $V(\ell)$ for some $\ell \in \mathbb{Z}_{\geq 0}$.
- We have the following Clebsch-Gordan type decomposition:

$$V(\ell) \otimes V(\ell') \cong \bigoplus_{j=|\ell-\ell'|}^{\ell+\ell'} V(j) \quad \text{(Scheunert-Nahm-Rittenberg 1977)}$$

Problems

- What about infinite-dimensional representations?
- How can we find explicit subrepresentations U_j of $V(\ell) \otimes V(\ell')$ such that $U_j \cong V(j)$?

Primitive Vectors

Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be a basic Lie superalgebra.

Definition

Category $\mathcal{O}$ is the full subcategory of $\mathrm{Rep}\,\mathfrak{g}$ consisting of finitely generated locally $\mathfrak{g}_+$-finite weight modules.

Definition

Let $V \in \mathrm{Rep}\,\mathfrak{g}$. The space of *primitive (or extremal) vectors* in V is

$$V^+ = \{v \in V \mid x.v = 0 \; \forall x \in \mathfrak{g}_+\}$$

Fact

If $V \in \mathcal{O}$ is completely reducible, and $\{v_i\}_i$ is a weight basis for V^+ then

$$V = \bigoplus_i V_i \qquad V_i = U(\mathfrak{g})v_i$$

and each V_i is an irreducible representation.

Extremal Projector (Asherova, Smirnov, Tolstoy, ... 1971–)

- $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ basic Lie superalgebra
- $\mathcal{C}$ a subcategory of Rep $\mathfrak{g}$
- functors $(-)^+, (-)_- : \mathcal{C} \to \mathsf{SVec}$:

$$V^+ = \{v \in V \mid \mathfrak{g}_+ v = 0\}$$
$$V_- = V/\mathfrak{g}_- V \qquad (\textit{coinvariants})$$

- Inclusion $\iota_V : V^+ \to V$ and projection $\pi_V : V \to V_-$ compose to

$$\mathrm{Q}_V : V^+ \to V_- \quad v \mapsto v + \mathfrak{g}_- V \qquad \rightsquigarrow \mathrm{Q} : (-)^+ \Rightarrow (-)_-$$

Definition (HW 2021)

An *extremal projector* P *for* $\mathfrak{g}$ *in* $\mathcal{C}$ is an inverse of Q. Then $P_V := \iota_V \circ \mathrm{P}_V \circ \pi_V$ is a linear map $V \to V$ for any $V \in \mathrm{Rep}\,\mathfrak{g}$, satisfying

$$\mathfrak{g}_+ P_V = 0 = P_V \mathfrak{g}_- \qquad P_V^2 = P_V \qquad P_V \circ \iota_V = \iota_V \qquad \pi_V \circ P_V = \pi_V$$

In particular: $P_V : V \to V^+$ is a linear projection.

Examples

Let $\mathcal{C} = \mathcal{C}(\mathfrak{g})$ be the subcategory of $\mathcal{O}$ of modules with support contained in $\{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in R_+ : \lambda(\alpha^\vee) \notin \mathbb{Z}\}$.

Theorem (Tolstoy 1985)

If $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ is a basic Lie superalgebra, then $\mathfrak{g}$ has an extremal projector in $\mathcal{C}$.

Example

For $\mathfrak{g} = \mathfrak{sl}(2)$:

$$P = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(h + \rho(h) + n)_n} f^n e^n$$

where $\rho(h) = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial.

Mickelsson's Step Algebra

(Mickelsson, van den Hombergh, Zhelobenko, Khoroshkin, Ogievetsky,...)

- $\mathfrak{g} \subset \mathfrak{G}$ reductive pair of fin-dim'l complex Lie (super)algebras
- $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ triangular decomposition (assume $\mathfrak{g}$ basic)
- $U = U(\mathfrak{G})$
- $I = U\mathfrak{g}_+$ left ideal
- $N = N_U(I) = \{u \in U \mid \mathfrak{g}_+ u \subset U\mathfrak{g}_+\}$ normalizer of I in U
- $S(\mathfrak{G}, \mathfrak{g}) = N/I$ Mickelsson's *step algebra* (1973)

Lemma

If V is a $U(\mathfrak{G})$-module then $V^+ = \{v \in V \mid \mathfrak{g}_+ v = 0\}$ is an $S(\mathfrak{G}, \mathfrak{g})$-module.

Proof.

For $u + I \in S(\mathfrak{G}, \mathfrak{g})$ and $v \in V^+$: $\mathfrak{g}_+ uv \subset U\mathfrak{g}_+ v = 0$. So $uv \in V^+$. □

Theorem (van den Hombergh 1975)

If V is a locally $\mathfrak{g}$-finite irreducible $U(\mathfrak{G})$-module, then V^+ is an irreducible $S(\mathfrak{G}, \mathfrak{g})$-module.

Difficulties with $S(\mathfrak{G}, \mathfrak{g})$

1. $S(\mathfrak{G}, \mathfrak{g})$ is not a finitely generated $\mathbb{C}$-algebra. How can one write down elements of $S(\mathfrak{G}, \mathfrak{g})$?
2. How can one effectively find relations among elements?

Remarkable Observation

(Zhelobenko 1985)

Let $V = U/I = U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{g}_+$, regarded as a left $\mathfrak{g}$-module. Then

$$V^+ = \{u + U(\mathfrak{G})\mathfrak{g}_+ \mid \mathfrak{g}_+ u \subset U(\mathfrak{g})\mathfrak{g}_+\} = N/I = S(\mathfrak{G}, \mathfrak{g}).$$

In other words, *the step algebra itself is the space of $\mathfrak{g}_+$-invariants in the universal relative Verma module U/I.*

Therefore, if we can use the extremal projector P we can describe $S(\mathfrak{G}, \mathfrak{g})$ and resolve the difficulties.

Zhelobenko's Reduction Algebra

To deploy P one replaces U by $U' = U(\mathfrak{G})[(h_\alpha - n)^{-1} \mid n \in \mathbb{Z}, \alpha \in \Phi(\mathfrak{g})]$ in the construction of $S(\mathfrak{G}, \mathfrak{g})$ to obtain

$$Z(\mathfrak{G}, \mathfrak{g}) = N_{U'}(I')/I', \quad I' = U'\mathfrak{g}_+$$

which is called the *reduction algebra* of the pair $\mathfrak{g} \subset \mathfrak{G}$. This ensures that U'/I' is an object of $\mathcal{C}$ so that we have

$$P_{U'/I'} : U'/I' \twoheadrightarrow (U'/I')^+ = Z(\mathfrak{G}, \mathfrak{g})$$

The following addresses the "difficulties":

Theorem

1. *Decompose $\mathfrak{G} = \mathfrak{g} \oplus \mathfrak{p}$ as $\mathfrak{g}$-modules. Then the image of $\mathfrak{p}$ in $Z(\mathfrak{G}, \mathfrak{g})$ generates $Z(\mathfrak{G}, \mathfrak{g})$ as a $U'(\mathfrak{h})$-ring. (Mickelsson 1973)*
2. *The bijection $\mathrm{Q}_{U'/I'} : Z(\mathfrak{G}, \mathfrak{g}) \to (U'/I')_- = \mathfrak{g}_- U' \backslash U'/U'\mathfrak{g}_+$ equips the double coset space with an associative product $\bar{u} \Diamond \bar{v} = \overline{uPv}$. (Khoroshkin-Ogievetsky 2008; HW 2021)*

Remarks

Theorem

Let V be an irreducible representation of $\mathfrak{G}$ such that $V \in \mathcal{C}(\mathfrak{g})$, then V^+ is an irreducible representation of $Z(\mathfrak{G}, \mathfrak{g})$.

Remark

$Z(\mathfrak{G}, \mathfrak{g}) \cong S(\mathfrak{G}, \mathfrak{g}) \otimes_{U(\mathfrak{h})} U'(\mathfrak{h})$

So if $z \in Z(\mathfrak{G}, \mathfrak{g})$ then $fz \in S(\mathfrak{G}, \mathfrak{g})$ for some $f \in U(\mathfrak{h}) \setminus \{0\}$.

Previous Work

The reduction algebras $Z(\mathfrak{G}, \mathfrak{g})$ have been studied extensively when $\operatorname{rk} \mathfrak{G} \leq 1 + \operatorname{rk} \mathfrak{g}$, including for

- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n)$ (van den Hombergh 1976; Zhelobenko 1983–)
- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(m|n), \mathfrak{osp}(n|2m)$, m fixed (Tolstoy 1986)
- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{so}(n), \mathfrak{so}(n-2)\big)$ and $\big(\mathfrak{sp}(2n), \mathfrak{sp}(2n-2)\big)$ (Molev 2000).

The quantum analog of the reduction algebra, $Z_q(\mathfrak{G}, \mathfrak{g})$, associated to $U_q(\mathfrak{g}) \subset U_q(\mathfrak{G})$ has also been studied for $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where

- $\mathfrak{g}(n) = \mathfrak{su}(n)$ (Tolstoy 1990)
- $\mathfrak{g}(n) = \mathfrak{su}(1|n)$ (Palev-Tolstoy 1991)
- $\mathfrak{g}(n) = \mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ (Ashton-Mudrov 2015)

Diagonal Reduction Algebras

Take $\mathfrak{G} = \mathfrak{g} \times \mathfrak{g}$. The diagonal embedding $\mathfrak{g} \subset \mathfrak{g} \times \mathfrak{g}$ gives rise to $DR(\mathfrak{g}) = Z(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ called the *diagonal reduction algebra of* $\mathfrak{g}$.

Application

If V and W are irreducible highest weight representations of $\mathfrak{g}$ such that $V \otimes W \in \mathcal{C}$, then $(V \otimes W)^+$ is an irreducible $DR(\mathfrak{g})$-module.

Theorem (Khoroshkin-Ogievetsky 2011, 2017)

For $\mathfrak{g} = \mathfrak{gl}(n)$*:*

1. *Complete presentations of* $DR(\mathfrak{g})$ *including one in terms of the reflection equation from R-matrix formalism.*
2. *Construction of* $2n$ *central elements of* $DR(\mathfrak{g})$ *that conjecturally generate the whole center.*
3. $DR(\mathfrak{g})$ *has the structure of a braided bialgebra.*

Main Result 1

Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. For $g \in \{f, y, h, x, e\}$ put

$$\bar{g} = P_{U'/I'}(g \otimes 1 - 1 \otimes g + I') \in DR(\mathfrak{g})$$

We know that $\{\bar{f}, \bar{y}, \bar{h}, \bar{x}, \bar{e}\}$ generates $DR(\mathfrak{g})$ as an $U'(\mathfrak{h})$-ring, where

$$U'(\mathfrak{h}) = \mathbb{C}[h][(h-n)^{-1} \mid n \in \mathbb{Z}]$$

Theorem (HW 2021)

1. *Complete presentation of $DR(\mathfrak{g})$. 12 relations, they look look like*

$$\bar{e}\bar{y} = r_1(h)\bar{y}\bar{e} + r_2(h)\bar{h}\bar{x}$$

   where $r_i(h)$ are rational functions of h.

2. *PBW type basis: $DR(\mathfrak{g})$ is a free left $U'(\mathfrak{h})$-module on the set*

$$\{\bar{f}^i \bar{y}^j \bar{h}^k \bar{x}^l \bar{e}^m \mid i, j, k, l, m \in \mathbb{Z}_{\geq 0}; j, l \leq 1\}$$

Algorithm in Proof

1. Pass to double coset space, compute all ordered quadratic monomials, for ex

$$\begin{aligned}\bar{y}\bar{e} &\mapsto (y\otimes 1 - 1\otimes y)P(e\otimes 1 - 1\otimes e) + \mathfrak{g}_- U + U\mathfrak{g}_+ \\ &= \sum_{n=0}^{\infty} \varphi_n(h\pm 1)(y\otimes 1 - 1\otimes y)y^n x^n(e\otimes 1 - 1\otimes e) + \mathfrak{g}_- U + U\mathfrak{g}_+ \\ &= \sum_{n=0}^{\infty} \varphi_n(h\pm 1)(-\operatorname{ad} y)^n(y\otimes 1 - 1\otimes y)(\operatorname{ad} x)^n(e\otimes 1 - 1\otimes e) \\ &\quad + \mathfrak{g}_- U + U\mathfrak{g}_+ = \dots\end{aligned}$$

2. Express in terms of ordered quadratic monomials in $U(\mathfrak{G})$.
3. Invert triangular system of linear equations.
4. For any mis-ordered product eg $\bar{e}\bar{y}$, compute it as above then use inverse system to write using ordered products.

Gorelik's Ghost Center

The *center* of an associative superalgebra A consists of all sums of homogeneous z satisfying

$$za = (-1)^{|z||a|}az$$

for all homogeneous $a \in A$.

Definition (Gorelik 2000)

1. The *anti-center* $\Sigma = \Sigma(A)$ of an associative superalgebra A is given by all sums of homogeneous z satisfying

   $$za = (-1)^{(|z|+\bar{1})|a|}az$$

   for all homogeneous $a \in A$.
2. The *ghost center* is $\mathbb{Z}(A) = Z(A) \oplus \Sigma(A)$.

Main Result 2: Ghost Center of $DR\big(\mathfrak{osp}(1|2)\big)$

Put

- $\mathfrak{g} = \mathfrak{osp}(1|2)$
- $C \in Z(U(\mathfrak{g}))$ the Casimir element
- $Q \in \Sigma\big(U(\mathfrak{g})\big)$ the Scasimir element (Leśniewski 1995)

Theorem (HW 2022)

Let $A = DR\big(\mathfrak{osp}(1|2)\big)$. The ghost center $\mathbb{Z}(A)$ is generated by the three elements

$$\mathbb{C}_\pm := C \otimes 1 \pm 1 \otimes C + I' \in Z(A),$$

$$\mathbb{Q} := Q \otimes Q + I' \in \Sigma(A).$$

Moreover, there is an injective algebra map

$$\varphi : \mathbb{Z}(A) \to \mathbb{C}[x, y]$$

such that $\varphi(\mathbb{C}_+) = x^2 + y^2, \quad \varphi(\mathbb{C}_-) = 2xy, \quad \varphi(\mathbb{Q}) = x^2 - y^2.$

Steps in Proof

1. Using our PBW basis, we define an analog of the Harish-Chandra homomorphism $\varphi : \mathfrak{X}(A) \to U'(\mathfrak{h})[\bar{h}] \subset \mathbb{C}(h, \bar{h})$.
2. irreducible Verma modules $\Rightarrow \ker \varphi = 0$
3. reducible Verma modules $\Rightarrow \operatorname{im} \varphi$ consists of relative $\mathbb{Z}^2 \times \mathbb{Z}^2$-invariants $\mathbb{C}[x, y]_{\chi}^{\mathbb{Z}_2 \times \mathbb{Z}_2}$, $\chi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \{\pm 1\}$ character
4. check that $\{\varphi(\mathbb{C}_+), \varphi(\mathbb{C}_-), \varphi(\mathbb{Q})\}$ generates $\operatorname{im} \varphi$.

Key technical lemma for step 3 is computing the radical of a Shapovalov type form on Verma modules over $DR(\mathfrak{g})$.

Main Result 3: Irreps of $DR\big(\mathfrak{osp}(1|2)\big)$

Theorem (HW 2022)

Let $A = DR\big(\mathfrak{osp}(1|2)\big)$.

1. *For every odd positive integer* n *and every* $(\lambda, \mu) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z})$ *satisfying*
$$\lambda^2 = (\mu + n)^2$$
there is an irreducible n*-dimensional representation* $L(\lambda, \mu)$ *of* A *such that the action of the ghost center on* $L(\lambda, \mu)$ *is given by*
$$\mathbb{C}_+ \mapsto \lambda^2 + \mu^2 \qquad \mathbb{C}_- \mapsto 2\lambda\mu \qquad \mathbb{Q} \mapsto (\lambda^2 - \mu^2)(-1)^{|\cdot|}$$
where $(-1)^{|\cdot|} \in \mathrm{End}_{\mathbb{C}}\big(L(\lambda, \mu)\big)$ *sends homogeneous* v *to* $(-1)^{|v|}v$.
2. *Every finite-dimensional irreducible representation of* A *has odd dimension and is isomorphic to* $L(\lambda, \mu)$ *for a unique pair* (λ, μ) *satisfying* $\lambda^2 = (\mu + \dim V)^2$.

Application to Tensor Product Decompositions

Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. For $\ell \in \mathbb{Z}_{\geq 0}$, let $V(\ell)$ be the $(1+2\ell)$-dimensional irrep of $\mathfrak{g}$, and $\mathbb{C}[x] = V(-1/2)$ be the polynomial irrep of $\mathfrak{g}$. We know:

$$V(\ell) \otimes V(\ell') \cong \bigoplus_{j=|\ell-\ell'|}^{\ell+\ell'} V(j) \quad \text{(Scheunert-Nahm-Rittenberg 1977)}$$

$$\mathbb{C}[x] \otimes V(1) \cong \bigoplus_{j=0}^{2} V(1 - \frac{1}{2} - j) \quad \text{(special case of Coulembier 2013)}$$

Theorem (HW 2022)

$$\mathbb{C}[x] \otimes V(\ell) = \bigoplus_{j=0}^{2\ell} U(\mathfrak{g}_-)\bar{y}^j \cdot (1 \otimes v_\ell) \cong \bigoplus_{j=0}^{2\ell} V(1 - \frac{1}{2} - j)$$

where $\bar{y} \in N \subset U(\mathfrak{g} \times \mathfrak{g})$ is a lowering operator explicitly given in a PBW basis for $\mathfrak{g} \times \mathfrak{g}$.

Future directions

- Can $DR\big(\mathfrak{osp}(1|2n)\big)$ be presented using R-matrix formalism, analogous to the reflection equation for $DR\big(\mathfrak{gl}(n)\big)$?
- One can define $Z(A,\mathfrak{g})$ where A is an associative superalgebra and $\mathfrak{g}\to A$. We are interested in $Z\big(A_n(\mathbb{C})\otimes U\big(\mathfrak{osp}(1|2n)\big),\mathfrak{osp}(1|2n)\big)$ and applications to intertwining operators for $\mathbb{C}[x_1,\ldots,x_n]\otimes V(\lambda)$.

References

1. arXiv:2106.04380 [math.RT] *Diagonal reduction algebra for* $\mathfrak{osp}(1|2)$ (in Theoretical and Mathematical Physics), with D.A. Williams II
2. arXiv:2203.08068 [math.RT] *Ghost center and representations of the diagonal reduction algebra of* $\mathfrak{osp}(1|2)$, with D.A. Williams II

Future directions

- Can $DR(\mathfrak{osp}(1|2n))$ be presented using R-matrix formalism, analogous to the reflection equation for $DR(\mathfrak{gl}(n))$?
- One can define $Z(A, \mathfrak{g})$ where A is an associative superalgebra and $\mathfrak{g} \to A$. We are interested in $Z\big(A_n(\mathbb{C}) \otimes U(\mathfrak{osp}(1|2n)), \mathfrak{osp}(1|2n)\big)$ and applications to intertwining operators for $\mathbb{C}[x_1, \ldots, x_n] \otimes V(\lambda)$.

References

1. arXiv:2106.04380 [math.RT] *Diagonal reduction algebra for* $\mathfrak{osp}(1|2)$ (in Theoretical and Mathematical Physics), with D.A. Williams II
2. arXiv:2203.08068 [math.RT] *Ghost center and representations of the diagonal reduction algebra of* $\mathfrak{osp}(1|2)$, with D.A. Williams II

Thank you for your attention.