Multigraded Stillman's Conjecture

UW Madison Algebra and Algebraic Geometry Seminar

John Cobb
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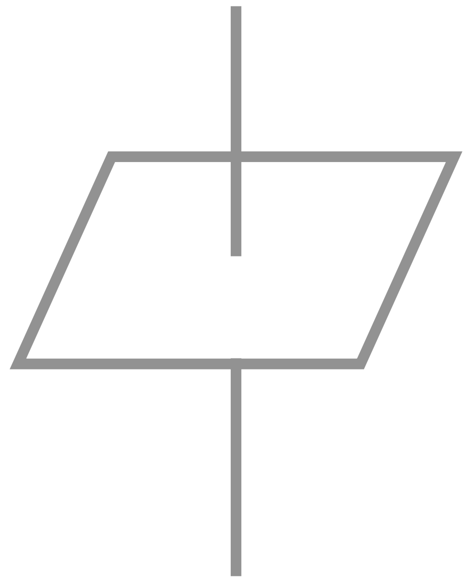
Syzygies

A *syzygy* is a polynomial relation among a collection of polynomials

Example. $S = k[x_0, x_1, x_2]$, and $I = \langle x_0x_1, x_0x_2 \rangle$

Goal: Study syzygies of S/I.



minimal free resolution

$$0 \longleftarrow S/I \xleftarrow{\;1\;} S \xleftarrow{(x_0x_1 \quad x_0x_2)} S(-2)^2 \xleftarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} S(-3) \longleftarrow 0$$

$\langle x_0x_1, x_0x_2 \rangle$ is the
module of 1st syzygies

they give relations
on the generator 1

$$(1)x_0x_1 = 0$$
$$(1)x_0x_2 = 0$$

$\left\langle \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\rangle$ is the
module of 2nd syzygies

they give relations
on the generators
x_0x_1 and x_0x_2

$$x_2(x_0x_1) - x_1(x_0x_2) = 0$$

Syzygies

Any finitely generated graded S-module M has a minimal free resolution:

$$0 \longleftarrow M \longleftarrow \bigoplus S(-j)^{\beta_{0,j}} \longleftarrow \bigoplus S(-j)^{\beta_{1,j}} \longleftarrow \cdots$$

This lists *all* information about M. If $M = S/I$, then the minimal free resolution *sees the geometry* of $V(I)$.

Problem: These are hard to find!

Moral of syzygies: All $\underbrace{\text{degrees and ranks}}_{\textit{easier to find}}$ of syzygies are numerical invariants of M!

Hilbert Syzygy Theorem and Stillman's Conjecture

Hilbert Syzygy Theorem (1890). If $I \subseteq k[x_0, \ldots, x_n]$ is finitely generated, the $\underbrace{\text{length of the minimal free resolution}}_{\textit{projective dimension}}$ of $k[x_0, \ldots, x_n]/I$ is bounded by $n+1$.

Stillman's Conjecture (2000). The projective dimension of $k[x_0, \ldots, x_n]/I$ can be bounded in terms of the number and degree of the generators of I, provided that $k[x_0, \ldots, x_n]$ is given the standard grading.

Important bit: This bound is independent of how many variables are in the polynomial ring!

Hilbert Syzygy Theorem and Stillman's Conjecture

Example. What is the projective dimension of

$$I = \langle x_1x_4x_7 + x_{10}x_{13}x_{16}\, , x_2x_5x_8 + x_{11}x_{14}x_{17}\, , x_3x_6x_9 + x_{12}x_{15}x_{18} \rangle?$$

We *could* embed this into $k[x_1, \ldots, x_{18}]$ and get the bound 18.

If we give each variable $x_1, \ldots, x_{18}$ degree 1, then I is generated by three cubics. If Stillman's conjecture is true, then we could get some bound from this information alone.

Stillmans conjecture is true and Mantero and McCullough proved that the bound for three cubics is 5.

The Stillman Story

2000 Stillman conjectures the Stillman conjecture

2016 First proof by Ananyan and Hochster in 2016, (re)introduced *strength*

2018 Bik, Draisma, Eggermont prove that actually any nontrivial Zariski-closed condition on tensors that is functorial in the underlying vector space is detected by strength

2019 Proven again by Erman, Sam, and Snowden

and then again by Draisma, Lasoń, and Leykin shortly after

2021 Erman, Sam, Snowden shows that "projective dimension" can be swapped out with many other things

The Ananyan-Hochster Principle

Definition. The ***strength*** of a homogeneous element f in a graded k-algebra R is

$$\text{str}(f) = \min_k \{ f = \sum_{i=1}^{k+1} g_i h_i \mid g_i, h_i \text{ homogeneous positive degree elements of } R \}$$

the ***collective strength*** of a family $\{f_i\}$ is the minimal strength of any nontrivial homogeneous k-linear combination.

Examples. (a) str$($something deg $1) = \infty$, elements of R_+^2 are finite strength

(b) str$(f) = 0$ exactly means f is reducible; e.g. str$(x^2 - y^2) = 0$

(c) The polynomial $\sum_{i=1}^n x_i y_i z_i$ in $k[x_i, y_i, z_i]$ with standard grading has strength n

Moral: Polynomials of sufficiently large strength are regular sequences.

The Ananyan-Hochster Principle

Moral: Polynomials of sufficiently large strength are regular sequences.

Note: If collective strength of $\langle f_i \rangle$ is infinite in a polynomial ring, then f_i form a regular sequence.

More precisely: Fix $d = (d_1, \ldots, d_r)$. There exists $M(d)$ such that if $f_1, \ldots, f_r \in k[X]$ are polynomials with degrees d with collective strength at least M, then $f_1, \ldots, f_r$ are a regular sequence

Proof Sketch: Suppose not….

(degree d)

$$\left.\begin{array}{l} \langle f_{1,1}, \ldots, f_{1,r} \rangle \in k[X_1] \\ \langle f_{2,1}, \ldots, f_{2,r} \rangle \in k[X_2] \\ \vdots \end{array}\right\} \text{unbounded collective strength but none are a regular sequence}$$

some sort of nice limiting object (more later) $\longrightarrow$ $\langle \mathbf{f}_1, \ldots, \mathbf{f}_r \rangle \in \mathbf{k}[X]$ $\Big\}$ *infinite* collective strength but not a regular sequence

The Quest for Multigraded Analogues

What if $k[x_1, \ldots, x_n]$ is not standard graded?

Increasing structure

Polynomials	**Graded Polynomials**	**Multigraded Polynomials**
	or $\mathbb{Z}$-graded polynomials	or $\mathbb{Z}^r$-graded polynomials
$5xy + y^2 - x^3$	$\underbrace{x_0^2 + 3x_1x_3 - 6x_2^2}_{\text{degree } = 2 \in \mathbb{Z}}$	$\underbrace{x_0^3y_1^2 - 2x_0x_1^2y_0y_1}_{\text{degree } = (3,2) \in \mathbb{Z}^2}$
affine geometry	projective geometry	toric geometry

[Haiman, Sturmfelds ’02] [Maclagan, Smith ’04] [Hering, Schenck, Smith ’06] [Costa, Miró-Roig ’06]
[Tai Há ’07] [Lozovanu, Smith ’12] [Yang ’19] [Berkesch, Erman, Smith ’20] [Chardin, Nemati ’20]
[Chardin, Holanda ’22] [Bruce, Heller, Sayrafi ’22] [Brown, Sayrafi ’22] [Booms, Cobb ’22] [Cobb ’24]

The Quest for Multigraded Analogues

Example. Let $S = k[x_0, x_1, y_0, y_1, y_2]$ be the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$ with the associated $\mathbb{Z}^2$-grading:

$$\deg(x_i) = (1,0) \text{ and } \deg(y_i) = (1,0)$$

The strength of any homogeneous polynomial in S is bounded by 2. This is NOT true in the standard graded case.

The Quest for Multigraded Analogues

Example. Let $S = k[x, y, z_1, \ldots, z_n]$ be a $\mathbb{Q}$-graded polynomial ring where all variables are given degree $1/n$. The homogeneous ideal

$$I = \langle x^n, y^n, x^{n-1}z_1 + x^{n-2}yz_2 + \cdots + xy^{n-2}z_{n-1} + y^{n-1}z_n \rangle$$

is generated by degree 1 elements.

McCullough showed that $\operatorname{pdim}_S(S/I) = n + 2$.

This is a counterexample to Stillman's conjecture (with this grading) because we can increase n above any potential bound.

So maybe we want to avoid having infinite decreasing sequences….

That is, we want the grading to be ***well founded***.

The Main Theorems

A ***grading*** of a polynomial ring $S = k[X]$ by an abelian group Γ is a decomposition of S into k-submodules $S = \bigoplus_{g \in \Gamma} S_g$ with $S_g \cdot S_h \subseteq S_{g \cdot h}$.

The ***support*** of Γ is a submonoid Λ generated by the degrees of all the monomials along with the identity. We'll say S is ***connected*** if $S_0 = k$.

Λ (or S) has ***bounded factorization*** if it is impossible to express an element in Λ in terms of arbitrarily large sums of other elements in Λ

bounded factorization $\Longrightarrow$ well founded

Fact. If S is connected and Λ is finitely generated then Λ has bounded factorization

The Main Theorems

Meta-Theorem (Cobb, Gallup, Spoerl). If Λ has bounded factorization, then the Ananyan-Hochster principle holds.

Theorem (Cobb, Gallup, Spoerl). For any degree sequence $d = (d_1, \ldots, d_n)$ from Λ, there is a number $N(\Lambda, d)$ bounding the projective dimension of any ideal with degree sequence bounded by d in any connected Γ-graded polynomial ring with support contained in Λ if and only if Λ has bounded factorization.

The Main Theorems

Theorem (Cobb, Gallup, Spoerl). For any degree sequence $d = (d_1, \ldots, d_n)$ from Λ, there is a number $N(\Lambda, d)$ bounding the projective dimension of any ideal with degree sequence bounded by d in any connected Γ-graded polynomial ring with support contained in Λ if and only if Λ has bounded factorization.

Is there any wiggle room here?

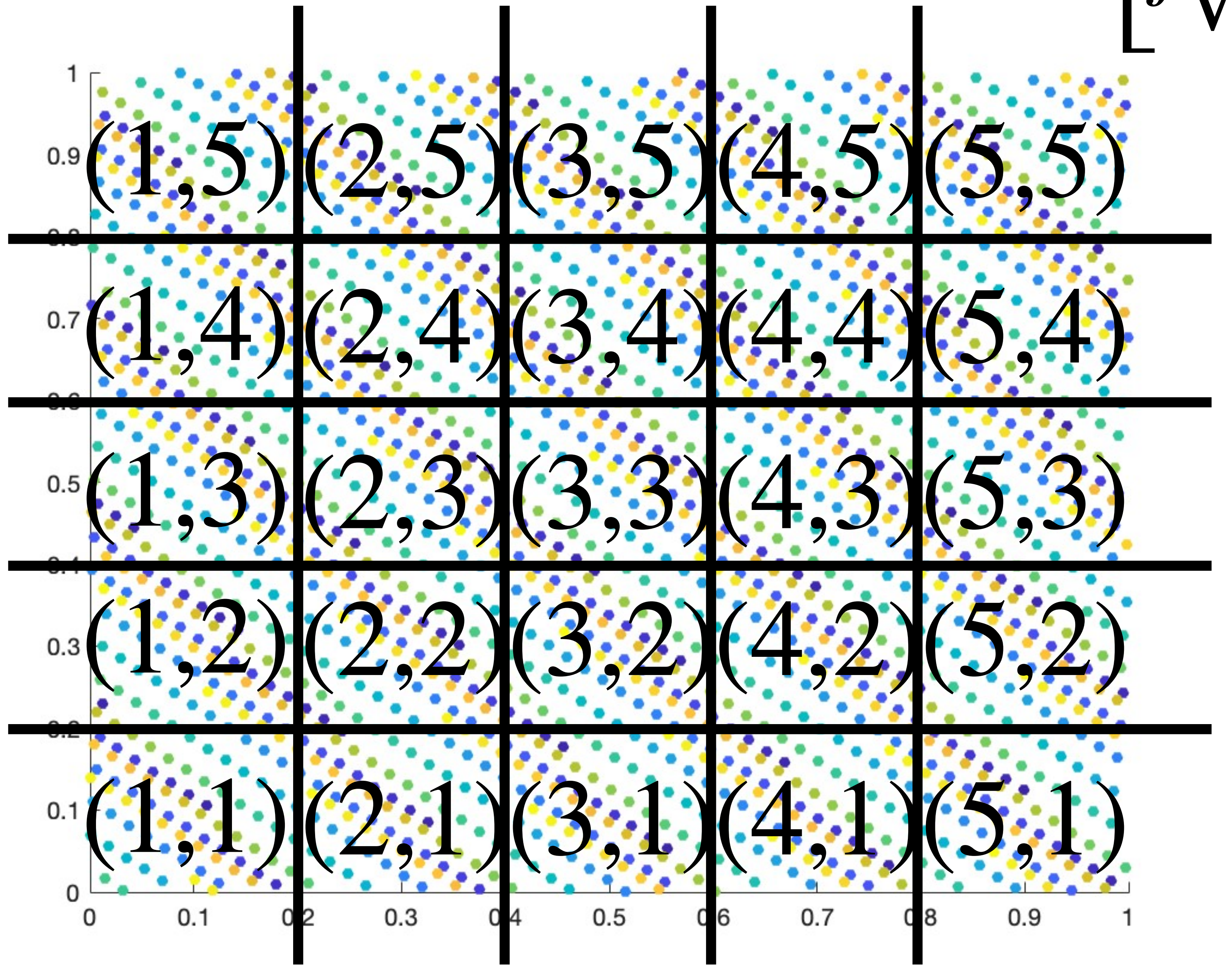
- The algebras *need* to be polynomial ring; it needs to be regular, but regular and graded implies polynomial.
- What about connectedness? Λ must necessarily be ***pointed*** ($q + q' = 0$ implies $q = q' = 0$), which is implied by connected. If $S_0 = k[x, y]$ then there is no Stillman bound. What about $S_0 = k[x]$? Polynomial ring over a PID?

The Main Theorems: Examples

Example. Fix an infinite polynomial ring $S = k[x_1, x_2, \ldots]$.

We will give it a $\mathbb{Z}^2$-grading associated with a choice of primes p, q.

First, note: The sequence $\begin{bmatrix} j\sqrt{p} & \mod 1 \\ j\sqrt{q} & \mod 1 \end{bmatrix}$ is equidistributed on the unit square.



Let $\deg(x_j)$ be the label of the bucket $\begin{bmatrix} j\sqrt{p} & \mod 1 \\ j\sqrt{q} & \mod 1 \end{bmatrix}$ lands in

For any p, q this grading has support generated by $(1,1)$, is connected, and has bounded factorization.

$\implies$ S has Stillman bounded projective dimension.

The Main Theorems: Examples

Example. What is the projective dimension of

$$I = \langle x_1x_4x_7 + x_{10}x_{13}x_{16}, x_2x_5x_8 + x_{11}x_{14}x_{17}, x_3x_6x_9 + x_{12}x_{15}x_{18}\rangle?$$

Mantero and McCullough proved that the bound for three cubics is 5.

Consider the $\mathbb{Z}^3$-grading:

$$\deg(x_j) = \begin{cases} (1,0,0) & \text{if } j = 1 \mod 3, \\ (0,1,0) & \text{if } j = 2 \mod 3, \\ (0,0,1) & \text{if } j = 0 \mod 3. \end{cases}$$

I now has degree sequence $e = ((3,0,0),(0,3,0),(0,0,3))$ and the Stillman bound $N(\Lambda, e)$ is 3.

Ultraproducts

Recall:

$\langle f_{1,1}, \ldots, f_{1,r} \rangle \in k[X_1]$

$\langle f_{2,1}, \ldots, f_{2,r} \rangle \in k[X_2]$

$\vdots$

unbounded collective strength but none are a regular sequence

Erman, Sam, Snowden do this, but do not use logic!

~~some sort of nice limiting object~~ $\longrightarrow$ $\langle \mathbf{f}_1, \ldots, \mathbf{f}_r \rangle \in \mathbf{k}[X]$

the ultraproduct

infinite collective strength but not a regular sequence

An ***ultraproduct*** **A** of a family $\{A_i\}_{i \in I}$ keeps track of generic properties of the family.

Less explicit than many other options, but it has extremely remarkable logical properties determining essentially all behavior.

(one of which guarantees that it's meaningful to take limits of arbitrary sequences of polynomials as above)

Ultraproducts and Model Theory

An ***ultraproduct*** **A** of a family $\{A_i\}_{i \in I}$ keeps track of generic properties of the family.

Less explicit than many other options, but it has extremely remarkable logical properties determining essentially all behavior.

1. **Łoś' Theorem:** First order properties of **A** are exactly those determined by the first order properties of A_i

   $\Longrightarrow$ Proof of Stillman's Conjecture: Being a regular sequence is first order.

2. **Expansion:** Ultraproducts behave well when you add new symbols to your language
3. **Saturation:** Ultraproducts contain "all limits"

Why use model theory? We can get a cleaner proof, are able to see what exactly is necessary for Stillman bounds, and can prove things you otherwise could not when working "by hand".

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2. **Expansion:** Ultraproducts behave well when you add new symbols to your language

3. **Saturation:** Ultraproducts contain "all limits" $\Longrightarrow$ As a set, **A** can very nasty

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Ultraproducts and Model Theory

The Problem

It would be nice, for example, if ultraproducts of polynomial rings were polynomial rings.

Examples. Let's take the family $\{k[x]\}_{i\in\mathbb{N}}$. The ultraproduct **R** is not close to being a polynomial ring. Consider the sequence of elements $\{1,x,x^2,\ldots,\}$ corresponds to an element **r** in the ultraproduct, but it must have infinite degree! In fact, **R** contains $\mathbf{k}[[x]]$ but is strictly worse.

We instead define the Λ***-bounded ultraproduct*** $\mathbf{A}_\Lambda$, which for graded algebras, is a substructure of **A** generated by homogeneous elements whose degree is less than some already existing degree in Λ.

Degree Bounded Ultraproducts

Theorem (Cobb, Gallup, Spoerl). A graded version of Łoś' Theorem, Expansion, and Saturation which holds for formulas with "degree bounded quantifiers".

Our definition is forced upon you if you wish your ultraproduct of graded algebras to still be a graded algebra:

Corollary (Cobb, Gallup, Spoerl). Let $(R_i)_{i\in I}$ be Λ-graded k-algebras. Then $\mathbf{R}_\Lambda$ is a $\boldsymbol{\Lambda}$-graded $\mathbf{k}$-algebra with $(\mathbf{R}_\Lambda)_g$ equal to the elements $\{r_i\}_{i\in I}$ where $\deg(r_i) = g$ almost everywhere.

What we buy with logic

Corollary (Cobb, Gallup, Spoerl). We have the following:

(a) $\mathbf{\Lambda}$ is well founded iff Λ has bounded factorization

(b) **k** is a field iff k_i is a field a.e. (+ some results about how characteristics carry over)

(c) So long as Λ has bounded factorization, $\mathbf{R}_\Lambda$ is a polynomial ring iff R_i are polynomial rings a.e.

(d) integral domain, reduced, irreducible, connectedness…..

Corollary (Cobb, Gallup, Spoerl). Fix a sequence R_i and $I_i \subseteq R_i$.

(a) **I** is an ideal iff $I_\bullet$ are ideals a.e.

(b) **I** is a prime ideal iff $I_\bullet$ are prime ideals a.e.

(c) **f** is a homogeneous element iff $f_\bullet$ are homogeneous elements of $R_\bullet$ of bounded degree a.e.

(d) Strength of **f** is finite iff strength of $f_\bullet$ bounded a.e.

(e) $\mathbf{f}_1, \ldots, \mathbf{f}_n$ is a regular sequence iff $f_{\bullet,1}, \ldots, f_{\bullet,n}$ is a regular sequence with uniformly bounded degree a.e.

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What we buy with logic

Corollary. I is a prime ideal iff $I_\bullet$ are prime ideals a.e.

Proof. Let I be any ideal. Due to the expansion property, we can add a predicate $I(x)$ to our language which is true when $x \in I$. The property of being prime can now be written:

$$\forall r, s \in R_+ \, [I(r \cdot s) \to (I(r) \vee I(s))]$$

This confirms that primeness is first order. Now, Łoś guarantees the iff above.

Open Questions

1. What other algebraic facts are easy from this perspective?

2. Degree bounded ultraproducts package up a common argument that is not restricted to Stillman's conjecture. What else can we do?

Thanks!!

A Crash Course in Model Theory

A *structure* $\mathscr{M}$ in a *language* $\mathscr{L}$ has a universe M and interpretations for:

- predicate symbols in $\mathscr{L}$ (this is a function $M \to \{\text{true, false}\}$)
- function symbols in $\mathscr{L}$ (this is a function $M \to M$)
- constant symbols in $\mathscr{L}$ (this is a 0-ary function)

Example. The language of ordered rings might have function symbols $\{+, -, \cdot\}$, predicate symbols $\{\leq\}$, and constant symbols $\{0,1\}$.

An $\mathscr{L}$-structure might be $M = \mathbb{R}$, with the obvious interpretation for the symbols above.

A Crash Course in Model Theory

We can then build *words* of $\mathscr{L}$ by concatenating symbols together with variable symbols $(x_1, x_2, \ldots)$ using logical connectives ($\vee$, $\wedge$, $=$, $\neg$) and quantifiers ($\exists, \forall$).

Formulas $\varphi(x_1, \ldots, x_n)$ are grammatical words. The notation $\mathscr{M} \models \varphi(s_1, \ldots, s_n)$ means that the formula $\varphi(x_1, \ldots, x_n)$ is *true* in $\mathscr{M}$ when each x_i is interpreted by the corresponding *parameter* $s_i \in M$.

Example. Consider the ordered ring structure $\mathscr{M}$ with universe $\mathbb{R}$ from before.

We might require that, for example, the sentence $(\forall x)[x \cdot 1 = 1 \cdot x = x]$ is true in $\mathscr{M}$.