Towards

Generalisations of algebraic structures in differential geometry

$C(n)$: associative superalgebra generated by $2n$ odd elements θ^a, E_a $(a=1,\dots,n)$ modulo

$$[\theta^a, \theta^b] = [E_a, E_b] = 0, \quad [E_a, \theta^b] = \delta_a{}^b$$

Lie superalgebras of Cartan type obtained as subalgebras of the commutator algebra of $C(n)$:

- $W(n) = \langle \theta^{a_1} \cdots \theta^{a_p} E_b \rangle$
- $S(n)$: subalgebra of $W(n)$ consisting of traceless elements:

$$\operatorname{tr}(\theta^{a_1} \cdots \theta^{a_p} E_b) = [\theta^{a_1} \cdots \theta^{a_p}, E_b]$$

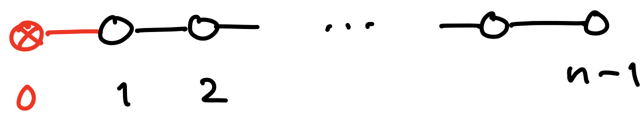
$\mathbb{Z}$-grading:

degree	basis of $W(n)$	basis of $sl(n\|1)$
1	E_a	E_a
0	$\theta^a E_b$	$K^a{}_b = \theta^a E_b$
-1	$\theta^a \theta^b E_c$	$F^a = \theta^a \theta^b E_b$
$\vdots$	$\vdots$	
$-n+1$	$\theta^{a_1} \cdots \theta^{a_n} E_b$	

Lie algebras appearing as subalgebras at degree zero:

$$W(n)_0 = gl(n)$$

$$S(n)_0 = sl(n) = A_{n-1}$$



Can be extended to a contragredient Lie superalgebra $A(0, n-1) = sl(n|1)$.

Question Given any Kac-Moody algebra $\mathfrak{g}$ and any contragredient Lie superalgebra extension B of $\mathfrak{g}$ "by a grey vertex", what is the natural generalisation W of $W(n)$? Is there an underlying generalisation C of the Clifford algebra $C(n)$?

The extension can be characterised by an integral dominant weight $\lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i$ where the λ_i are non-negative integers.

The grey vertex is connected to vertex j with λ_j lines.

As a $\mathfrak{g}$-module, $B_{-1} = R(\lambda)$, irreducible with highest weight λ.

Simplify: restrict to cases where $\mathfrak{g}$ is finite (any simple finite-dimensional Lie algebra) and $\lambda = \Lambda_j$ such that the corresponding Coxeter label is equal to one.

("pseudo-minuscule weight")

A_r:

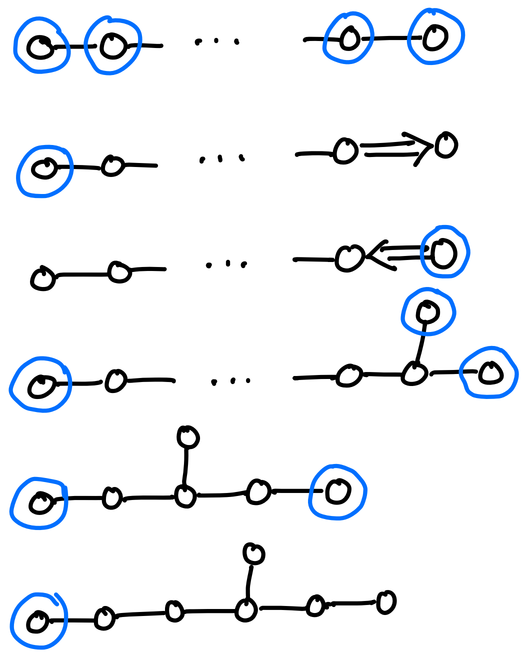
B_r:

C_r:

D_r:

E_6:

E_7:



(No such λ for E_8, F_4, G_2.)

The case $(g, \lambda) = (A_{n-1}, \Lambda_1)$:

degree	basis of $sl(n\|1)$:
1	E_a
0	$K^a{}_b = \theta^a E_b$
-1	$F^a = \theta^a \theta^b E_b$

With $F^a = \theta^a$, the bracket in $sl(n|1)$ differs from the commutator in $C(n)$:

$$[\![E_a, F^b]\!] = - F^b E_a + \delta_a{}^b F^c E_c$$

$$[E_a, F^b] = \delta_a{}^b$$

From $gl(n|1)$, with $K = F^a E_a$, we can reconstruct the products $F^b E_a$ and $E_a F^b$ in $C(n)$ by

$$F^b E_a = - [\![E_a, F^b]\!] + \delta_a{}^b K$$

$$E_a F^b = [\![E_a, F^b]\!] - \delta_a{}^b K + \delta_a{}^b$$

The general case (g, λ):

degree	basis of B	g-modules
...	...	...
1	E_M	$\overline{R(\lambda)}$
0	T_α, L	$g \oplus \mathbb{C}$
-1	F^M	$R(\lambda)$
...	...	...

$$\left(\begin{array}{l} M = 1, \ldots, \dim R(\lambda) \\ \alpha = 1, \ldots, \dim g \end{array} \right)$$

Grading element: $[L, B_k] = k B_k$

g-invariant bilinear form:

$$\langle E_M, F^N \rangle = - \langle F^N, E_M \rangle = \delta_M{}^N$$

As a generalisation of $C(n)$, we want a unital algebra with subspaces B_k $(k = 0, \pm 1)$ such that

$$F^N E_M = - [\![E_M, F^N]\!] - \delta_M{}^N L$$

$$E_M F^N = [\![E_M, F^N]\!] + \delta_M{}^N L + \delta_M{}^N$$

$\longrightarrow$

→ and, in addition,

$$[x_0, y_{\pm 1}] = [\![x_0, y_{\pm 1}]\!]$$

for $x_0 \in B_0$ and $y_{\pm 1} \in B_{\pm 1}$.

The only case where there is an associative such algebra is $(g, \lambda) = (A_{n-1}, \Lambda_1)$. In all other cases, we need to restrict associativity!

In the general case, there is a $\mathbb{Z}$-graded algebra $\tilde{C} = \tilde{C}_- \oplus \tilde{C}_0 \oplus \tilde{C}_+$ such that

- $\tilde{C}_{\pm k} \simeq U(B_0) \otimes T^k(B_{\pm 1})$
- the relations above hold for $B_k \subseteq \tilde{C}_k$ $(k = 0, \pm 1)$,
- $(XY)Z = X(YZ)$ whenever $X, Y \in \tilde{C}_{0\pm}$ or $Y, Z \in \tilde{C}_{0\pm}$.
- $(U \otimes 1)(V \otimes Y) = UV \otimes Y$
- $(U \otimes X)(1 \otimes Y) = U \otimes XY$

$(U, V \in U(B_0)$, $X, Y \in T(B_{\pm 1}))$

Non-associativity:

$$\begin{aligned}
(E_a F^b)E_c - E_a(F^b E_c) &= \\
&= \delta_a{}^b E_c - K^b{}_a E_c - E_a K^b{}_c = \\
&= -K^b{}_a E_c - K^b{}_c E_a = \\
&= -F^b(E_a E_c + E_c E_a)
\end{aligned}$$

Can we get a generalised Clifford algebra C by factoring out from $\tilde{C}$ the ideal generated by $\overline{R(2\lambda)} \subseteq \tilde{C}_2$, $R(2\lambda) \subseteq \tilde{C}_{-2}$?

"Cartanification":

There is furthermore a unique $\mathbb{Z}$-graded Lie superalgebra W and a surjective homomorphism

$$\varphi : B_{-1}B_0 \oplus B_0 \oplus B_1 \to W_{-1} \oplus W_0 \oplus W_1$$

such that

- W is nontrivial
- W is generated by $W_{-1} \oplus W_0 \oplus W_1$
- W is bitransitive: $(k \geqslant 0)$
  $[W_{-1}, w_k] = 0 \Rightarrow w_k = 0$
  $[W_1, w_{-k}] = 0 \Rightarrow w_{-k} = 0$
- $\varphi([x_i, y_j]) = [\varphi(x_i), \varphi(y_j)]$
  for $i+j = 0, \pm 1$ where
  $x_i, y_j \in B_{-1}B_0 \oplus B_0 \oplus B_1 \subset \tilde{C}$.

Ex $(\mathfrak{g}, \lambda) = (A_{n-1}, \Lambda_1)$:

$$B_1 = \langle E_a \rangle$$
$$B_0 = \langle F^a E_b \rangle$$
$$B_{-1}B_0 = \langle F^a K^b{}_c \rangle$$

φ must be injective on $B_0 \oplus B_1$.

What is $\varphi(F^a F^b E_c)$?

$$[E_a, F^b K^c{}_d] =$$
$$= [E_a, F^b] K^c{}_d$$
$$- F^b [E_a, K^c{}_d] =$$
$$= \delta_a{}^b K^c{}_d - \delta_a{}^c F^b E_d =$$
$$= \delta_a{}^b K^c{}_d - \delta_a{}^c K^b{}_d$$

For bitransitivity, we need

$$\varphi(F^a K^b{}_c) = -\varphi(F^b K^a{}_c)$$

Set $\varphi(x) = \bar{x}$.

$$[[\bar{E}_a, \bar{E}_b], \overline{F^c K^d{}_e}] =$$
$$= 2[\bar{E}_{(a}, [\bar{E}_{b)}, \overline{F^c K^d{}_e}]] =$$
$$= 4\,\delta_{(b}{}^{[c}[\bar{E}_{a)}, \overline{K^{d]}{}_e}] =$$
$$= 4\,\delta_{(b}{}^{[c}\delta_{a)}{}^{d]}\,\overline{K_e} = 0$$

Bitransitivity gives $[\bar{E}_a, \bar{E}_b] = 0$, and the full structure of $W(n)$.

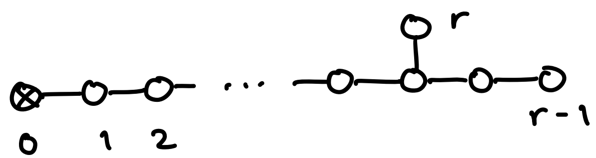
By restricting $B_0 = \mathfrak{g} \oplus \langle L \rangle$ to $\mathfrak{g}$ and $B_{-1}B_0$ to the maximal subspace $(B_{-1}B_0)_{\mathfrak{g}}$ such that $[B_1, (B_{-1}B_0)_{\mathfrak{g}}] \subseteq \mathfrak{g}$ we get similarly a $\mathbb{Z}$-graded Lie superalgebra S, such that $S = S(n)$ for $(\mathfrak{g}, \lambda) = (A_{n-1}, \Lambda_1)$.

The Lie superalgebras W and S are called tensor hierarchy algebras!

Ex) $(\mathfrak{g}, \lambda) = (D_r, \Lambda_1)$:

$$W = K(1|2r) \qquad S = H(2r)$$

Ex) $(\mathfrak{g}, \lambda) = (E_r, \Lambda_1)$: $(r \leq 7)$



Both W and S infinite-dimensional!

In S, there is a symmetry around degree $(9-r)/2$:

degree	$S(E_7, \Lambda_1)$	$B(E_7, \Lambda_1)$
...	...	...
3	912	912
2	133	133
1	56	56
0	133	$133 \oplus 1$
-1	912	56
-2	$133 \oplus 8645$	133
...	...	...

Let $\mathcal{W}$ be the Weyl algebra on $2(\dim R(\lambda))$ even elements p_M, x^N:

$$[x^M, x^N] = [p_M, p_N] = 0, \quad [p_M, x^N] = \delta_M{}^N$$

On the tensor product $\mathcal{W} \otimes \tilde{C}$, let d be an operator defined by

$$dx^M = F^M, \quad dE_M = p_M, \quad dF^M = dp_M = 0$$

on the generators, and extend it so that it acts as a derivation on $\mathcal{W}$, $\tilde{C}$ and $\mathcal{W}\tilde{C} \subseteq \mathcal{W} \otimes \tilde{C}$.

Define a vector field as an element in the subspace $\langle x^{M_1} \cdots x^{M_p} E_N \rangle$ of $\mathcal{W} \otimes \tilde{C}$.

For $(g, \lambda) = (A_{n-1}, \Lambda_1)$ we then get the ordinary Lie derivative of a vector field V with respect to a vector field U as a derived bracket:

$$
\begin{aligned}
[dU, V] &= [d(U^a E_a), V^b E_b] = \\
&= [dU^a E_a + U^a dE_a, V^b E_b] = \\
&= [\partial_c U^a F^c E_a + U^a p_a, V^b E_b] = \\
&= \partial_c U^a V^b [F^c E_a, E_b] \\
&\quad + U^a [p_a, V^b] E_b = \\
&= \partial_b U^a V^b E_a + U^a \partial_a V^b E_b \\
&= (\partial_a U^b V^a + U^a \partial_a V^b) E_b
\end{aligned}
$$

In the case $(g, \lambda) = (E_r, \Lambda_1)$, the derived bracket $[dU, V]$ describes a generalised Lie derivative which unifies r of the ordinary diffeomorphisms in 11-dimensional supergravity with gauge transformations.

Tensor hierarchy algebras can be defined also when λ is not a pseudo-minuscule weight, and even when $\mathfrak{g}$ is infinite-dimensional, by constructions different from the "cartanification" outlined here.

In general W_0 and W_1 are then strictly bigger than B_0 and B_1 !

Outlook

- How can the "cartanification" be modified so that it gives the expected result also when $\mathfrak{g}$ is infinite-dimensional ?
- Does the generalised Clifford algebra C exist ?

References

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