The Geometry and Combinatorics of Complex Polynomials

Jon McCammond (UC Santa Barbara)

Iowa State Univeristy
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Three Spaces

There are natural correspondences between three spaces:

- complex polynomials of degree d
- d-sheeted branched covers of a Euclidean rectangle R
- metric cell complexes built from noncrossing partitions

The third viewpoint leads to a new geometric combinatorial parameterization of the space of complex polynomials.

All new results are joint wtih Michael Dougherty. They come from our study of curvature properties of classifying spaces for braid groups, but might be of independent interest.

A Degree 5 Example

I'll begin with an example. Let p be the unique monic complex polynomial of degree 5 with $p(0) = 0$ and critical points

$$\mathbf{cpt}(p) = \{-\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{7-7i}{5}, \tfrac{10+i}{5}\}.$$

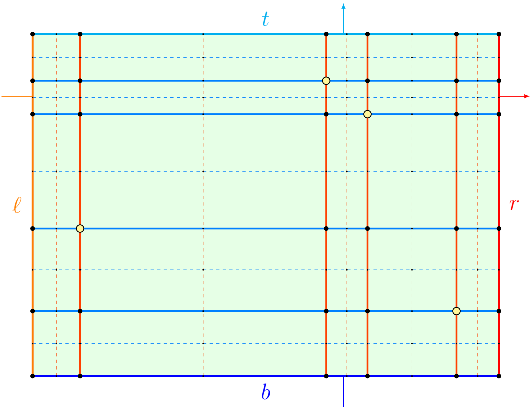
Concretely, $p(z)$ is the polynomial:

$$z^5 + \left(\tfrac{-17+6i}{4}\right) z^4 + \left(\tfrac{73-63i}{15}\right) z^3 + \left(\tfrac{34-12i}{25}\right) z^2 + \left(\tfrac{-308+252i}{125}\right) z.$$

The (rounded) critical values are

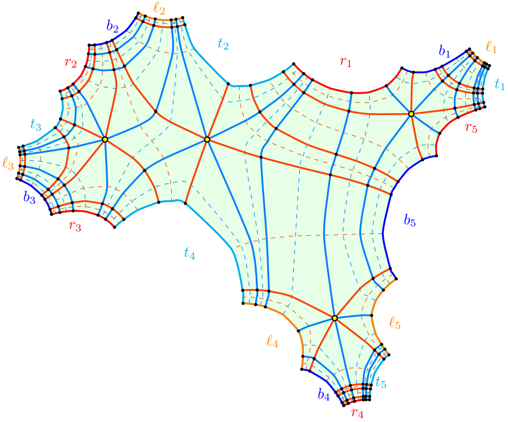
$$\mathbf{cvl}(p) \cong \{.8 - .6i, -.6 + .5i, -8.5 - 4.3i, 3.6 - 6.9i\}.$$

Subdivided Rectangle



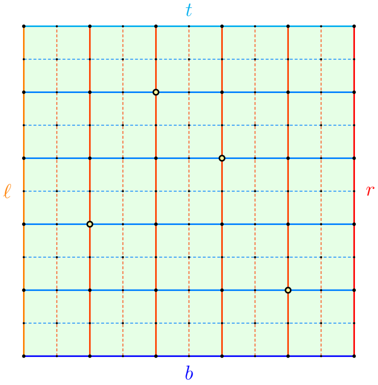
Given a subdivided rectangular complex R containing $\mathbf{cvl}(p)$...

Preimage Branched Rectangle



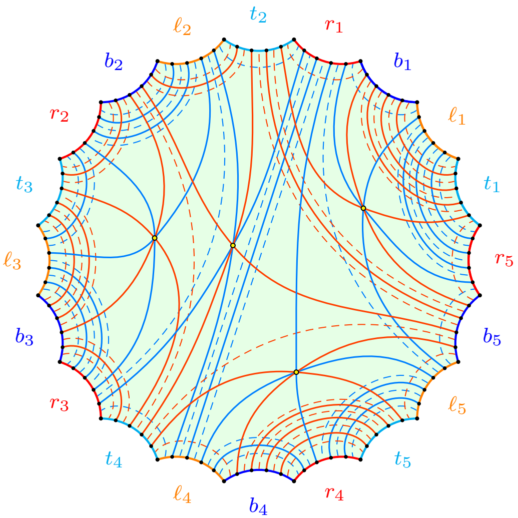
...the preimage $p^{-1}(R)$ is a branched rectangular complex.

Stylized Rectangle

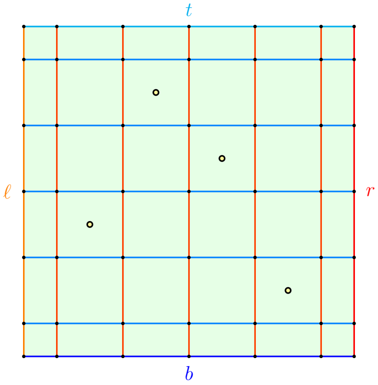


This stylized version suppresses the metric information to highlight the combinatorical structure.

Stylized Branched Rectangle

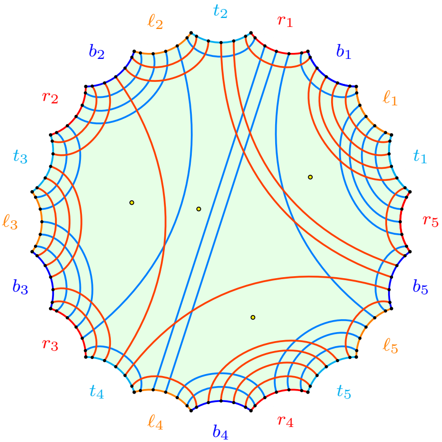


Dual Stylized Rectangle



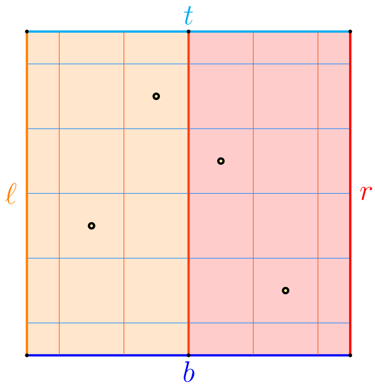
In this stylized cellular dual, the dashed curves and lines in the previous figure are the solid lines shown here.

Dual Stylized Branched Rectangle



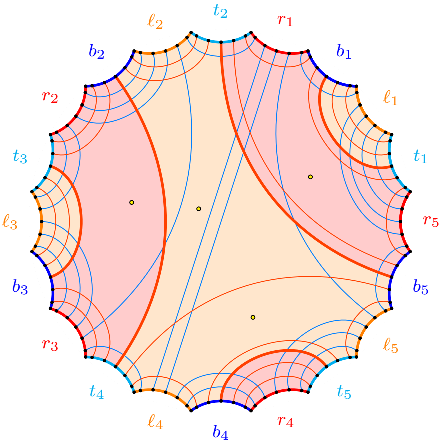
Since the lines avoid the critical values, their preimages are unions of disjoint unbranched arcs.

Regular Top-Bottom Split



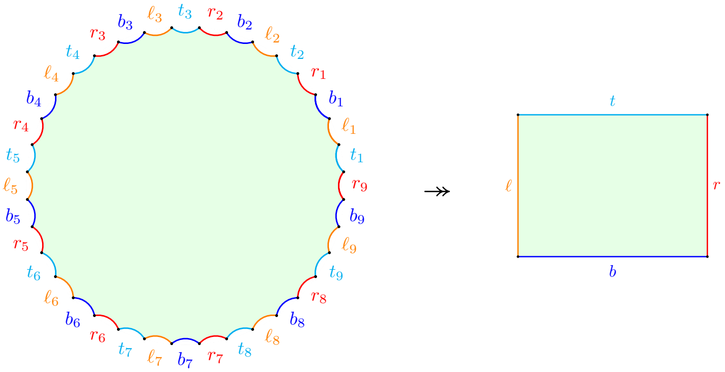
A vertically separated 2-coloring that avoids **cvl**(p) leads to ...

Top-Bottom Noncrossing Matchings



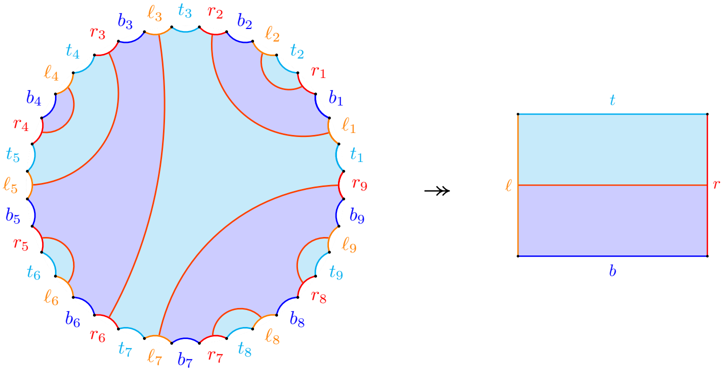
... 2-coloring of the preimage separated by a noncrossing matching of the d "top" sides and d "bottom" sides.

Stylized View of a Branched Rectangle



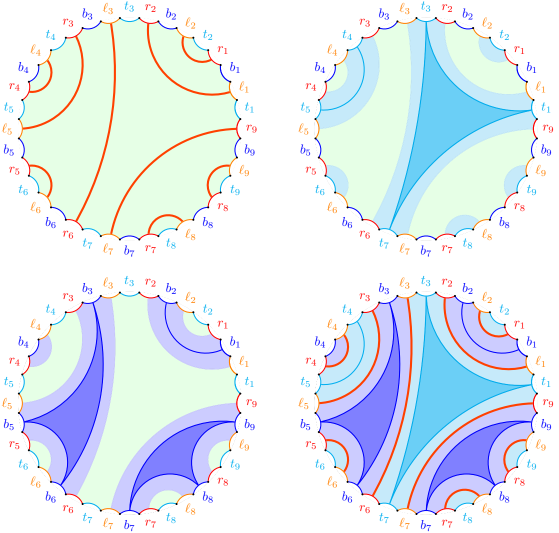
A *d*-sheeted branched rectangle, with the pullback metric, is a right-angled 4*d*-gon.

Line Preimages are NC Matchings

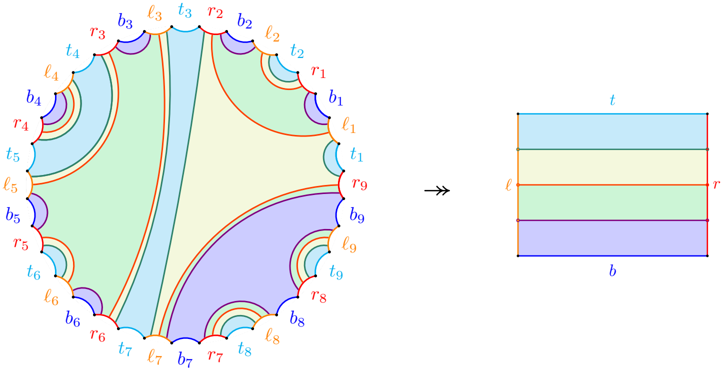


Preimages of regular lines are noncrossing matchings.

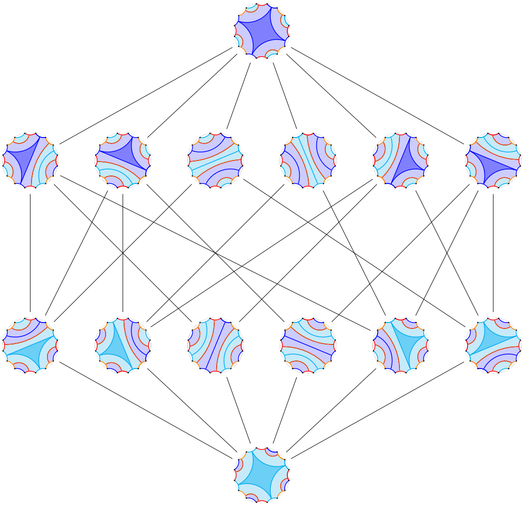
NC Matchings correspond to NC Partitions



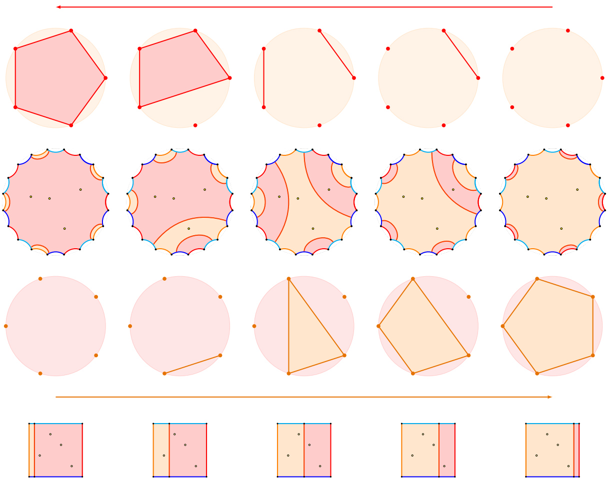
Parallel Line Preimages are NC Partition Chains



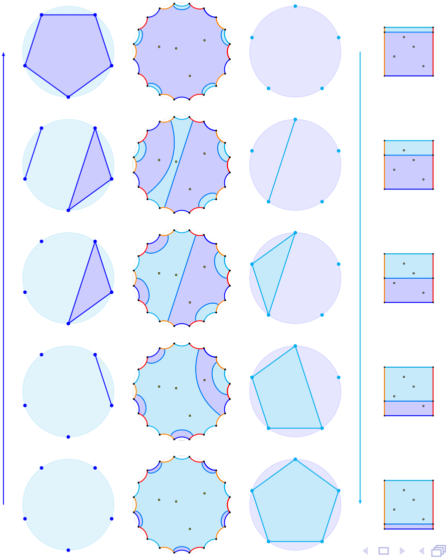
NC Partition Lattice



Left (or Right) NC Partitions in our Example



Top (or Bottom) NC Partitions in our Example



Gauss, FT of Algebra and Basketballs

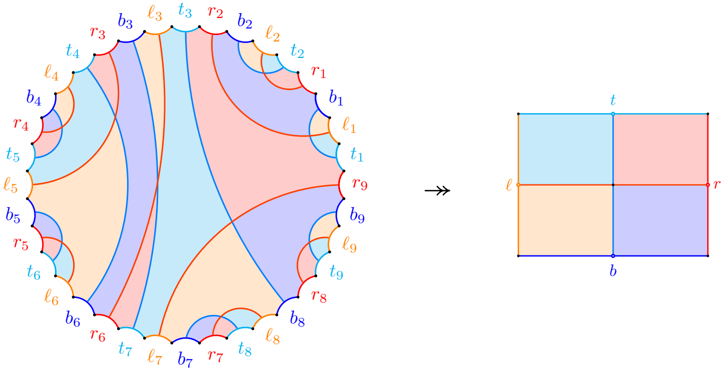
In Gauss' 1799 thesis he tried to prove the Fundamental Theorem of Algebra by focusing on the pullback of the real and imaginary axes.

These types of preimages were investigated more recently by Martin, Salvitt and Singer. They called them **basketballs**.

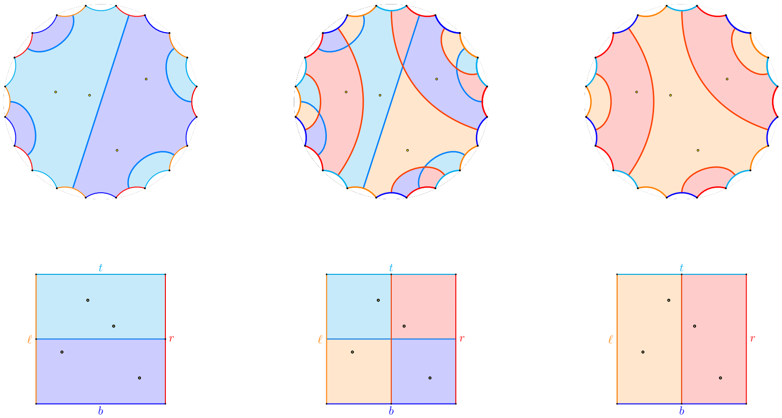
Definition

A **basketball** is a left-right NC matching and a top-bottom NC matching so that every l-r arc intersects exactly one t-b arc and every t-b arc intersects exactly one l-r arc.

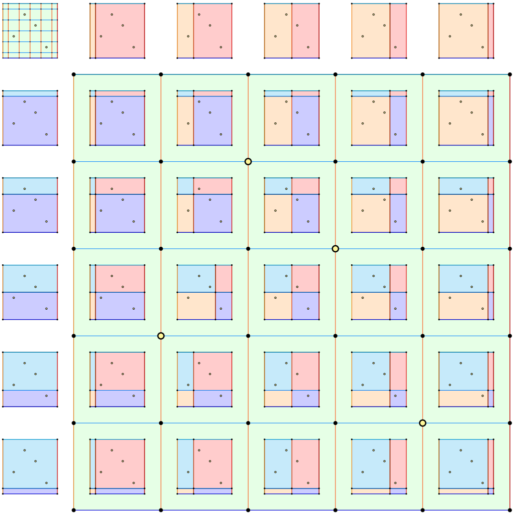
Crossing Line Preimages are Basketballs



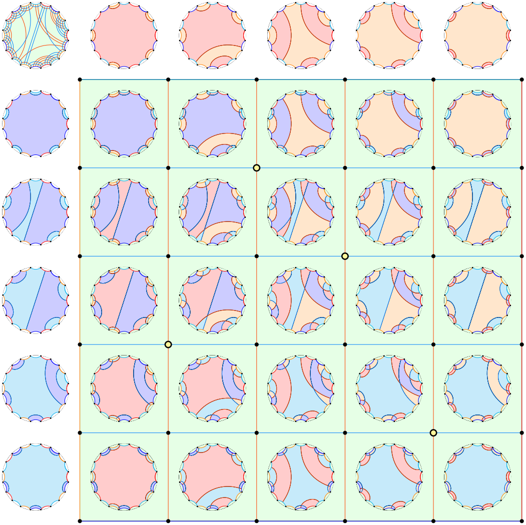
A Basketball in our Example



The 25 possible "plus signs" in our Example

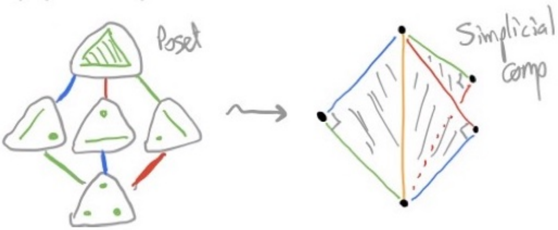


The 25 possible basketballs in our Example



Order Complex of NCPART$_d$

Let $|\text{NCPART}_d|$ be the order complex of the noncrossing partition lattice, i.e. the simplical complex whose simplices are indexed by chains in the poset. Its top-dimensional cells are n-dimensional simplices (where $n = d - 1$).



The noncrossing partition lattice for $d = 3$ and its order complex.

Basketball Vertices

The cell complex $|\mathrm{NCPART}_d| \times |\mathrm{NCPART}_d|$ has top-dimensional cells that are the product of two n-dimensional simplices.

Definition (Basketball vertices)

Vertices in the direct product are ordered pairs of vertices, one from each factor. If we interpret a vertex in the first factor as a left/right NC Matching (rather than a noncrossing partition) and a vertex in the second factor as a top/bottom NC Matching, then we can ask whether a vertex in the direct product corresponds to a basketball. If it does, this ordered pair is a **basketball vertex**.

The Basketball Complex

The vertices of $|\text{NCPART}_d| \times |\text{NCPART}_d|$ can be thought of as a NC top-bottom matching and a NC left-right matching.

Definition

The **basketball complex** is the full subcomplex restricted to the basketball vertices. We also call this BRRECT_d, the **branched rectangle complex**.

Remark

The full direct product is not a manifold, but the subcomplex BRRECT_d is a manifold (with boundary). In fact, a $2n$-ball.

Our Example and its 8-dimensional cell

The two chains determined by the combinatorics of a branched rectangle correspond to a cell in the basketball complex.

Remark (Branched Rectangles and cells)

In our example, we found two maximal chains in NCPART_5, one from the bottom to top Morse theory, and the other from the left to right Morse theory. And all 25 combinations of a noncrossing matching from each factor was a basketball. The maximal chain in each NCPART_5 determines a 4-simplex in the order complex and the two chains determine an 8-cell which is a product of two 4-simplices.

The Branched Rectangle Complex

Remark (Coordinates)

The combinatorics of a branched rectangle determines a cell in the basketball complex. And the metrics of the branched rectangle determine a specific point in the interior of this cell. In particular, the relative sizes of the widths of the small rectangles give barycentric coordinates in one simplex and the relative sizes of the heights of the small rectangles give barycentric coordinates in the other simplex.

Theorem (Basketballs and Branched Rectangles)

The points of the basketball complex BrRect_d *are in natural bijection with the space of all (based) planar* d*-sheeted metric branched covers of a metric rectangle, hence the name.*

From Branched Rectangles to Multisets

Given a based planar d-sheeted metric branched rectangle, the vertices which are “critical points” are sent to vertices in the range, which come equiped with multiplicities. In particular, there is a map from d-sheeted metric planar branched rectangles to n-element multisets in the rectangle R (with $n = d - 1$).

The next step is to put a cell structure on $\text{MULT}_n(R)$, the space of n-element multisets in R.

Polynomials = Branched Rectangles

Definition (Spaces of Polynomials)

Let $\textsc{Poly}_d^{mt}(U)$ be the space of monic complex polynomials of degree d with critical values in U up to precomposition with a translation.

Theorem

There are homeomorphisms $\textsc{BrRect}_d(R) \cong \textsc{Poly}_d^{mt}(R) \cong \mathbb{D}^{2n}$ *which restrict to* $\mathrm{int}(\textsc{BrRect}_d(R)) \cong \textsc{Poly}_d^{mt}(\mathrm{int}(R)) \cong \mathrm{int}(\mathbb{D}^{2n})$.

Theorem

Since $\textsc{Poly}_d^{mt}(\mathbb{C}) \cong \textsc{Poly}_d^{mt}(\mathrm{int}(R))$, *we get that the branched rectangle complex, built out of basketball vertices is a compactification of the monic polynomials up to translation.*

Our Main Theorems

The proof starts by mapping each polynomial to a point in the branched rectangle complex, then showing that it is onto the interior and finally showing that it's one-to-one.

Our Main Theorems

Theorem

$\textsc{Poly}_d(\mathbb{C}) \cong \text{int}(\textsc{BrRect}_d)$.
$\textsc{Poly}_d(\mathbb{C}_\mathbf{0}) \cong \text{int}(\textsc{BrAnn}_d)$.

The first focuses on the real and imaginary parts of the critical values. The second focuses on the magnitude and argument of the (ncessarily nonzero) critical values.

Remark

This embeds the dual braid complex (= $\textsc{BrCirc}_d$) in (the symmetric quotient of) the hyperplane complement, and the deformation retracts from $\textsc{BrAnn}_d$ to $\textsc{BrCirc}_d$ shows that the embedding is a homotopy equivalence.

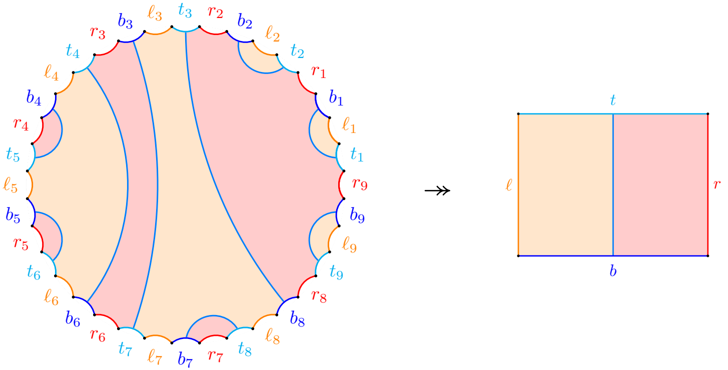
Branched Annuli

The branched annulus complex is constructed from the branched rectangle complex.

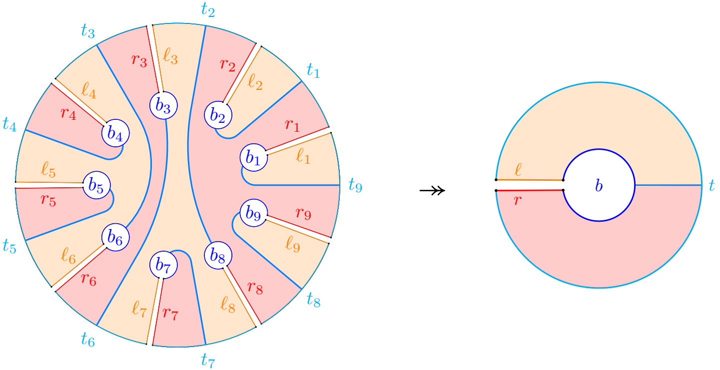
Definition

Every branched annulus can be viewed as a branched rectangle with identifications. In particular, the left sides are identified with the right sides (in a planar fashion) so that each bottom side become a circle and all of the top sides form the boundary of a disk with disks removed.

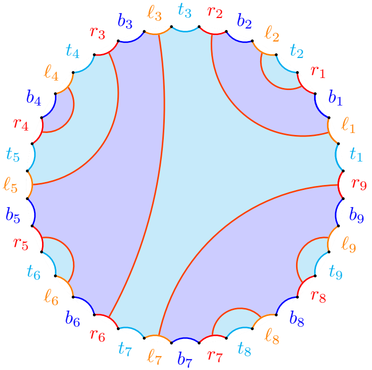
Rectangular Version of a TB Matching



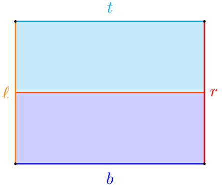
Annular Version of a TB Matching



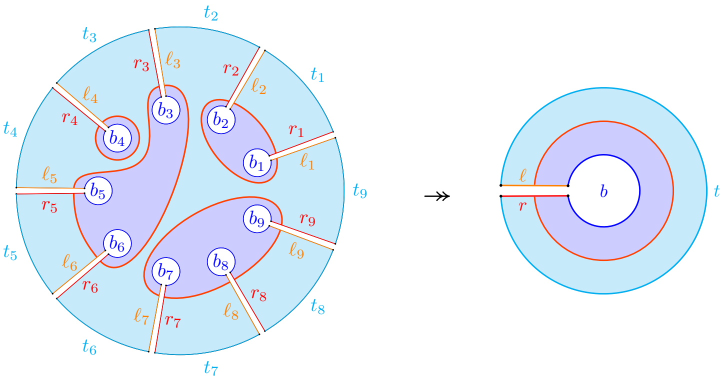
Rectangular Version of a LR Matching



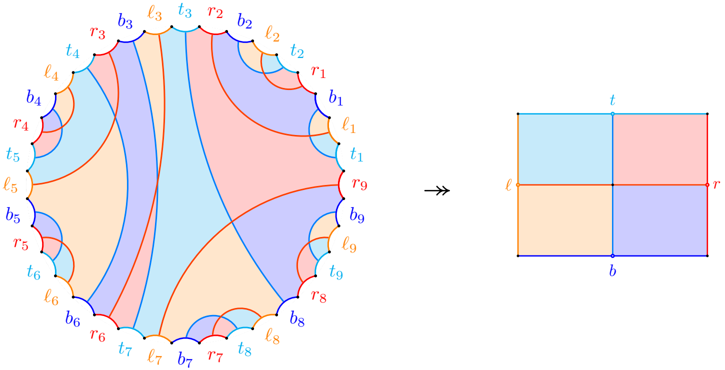
$\twoheadrightarrow$



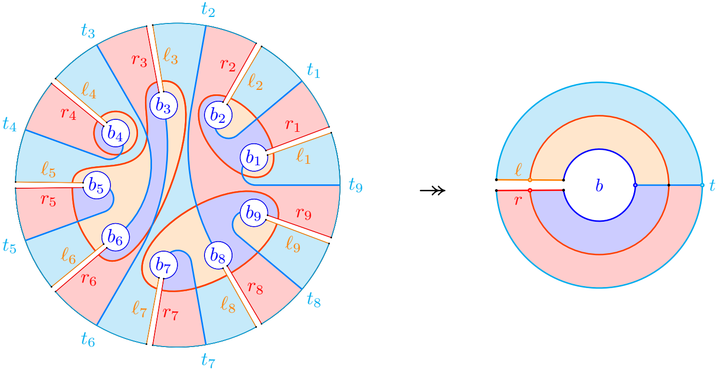
Annular Version of a LR Matching



Rectangular Version of a Basketball



Annular Version of a Basketball



Roots, Critical Points and Critical Values

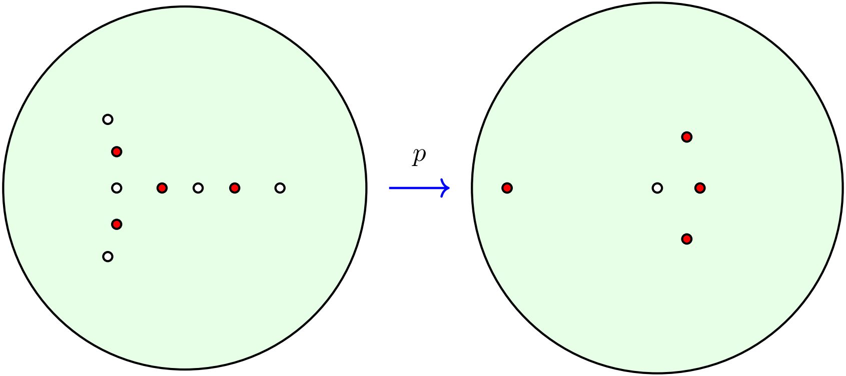
Let's focus on a single example, to illustrate the connection. Let $p(z)$ be a degree-d complex polynomial. The roots and critical points of $p(z)$ are in the domain. The critical values of $p(z)$ are in the range.

Lemma

For a polynomial $p(z)$, the following are equivalent:

- *p has no repeated roots,*
- *the roots and critical points are disjoint sets, and*
- *the critical values of p are nonzero.*

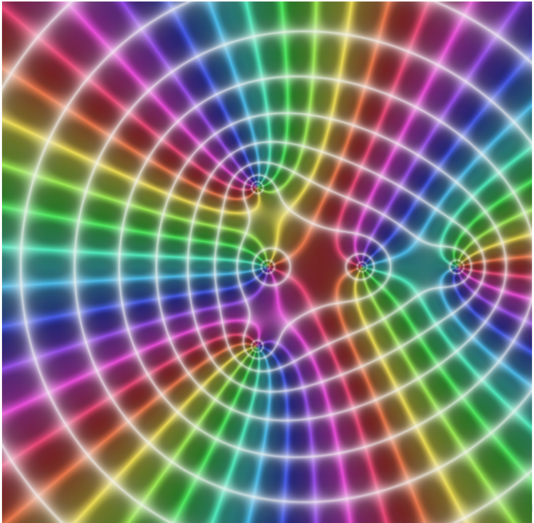
An Example: Roots 1



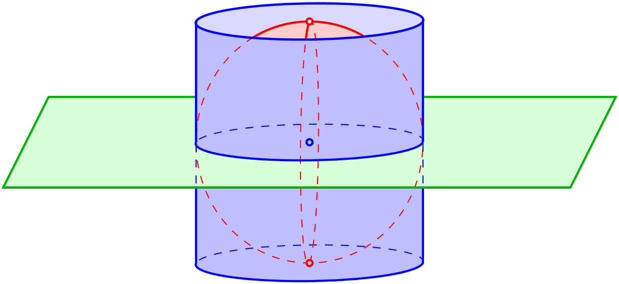
$$p(z) = 3z^5 - 15z^4 + 20z^3 - 30z^2 + 45z$$
$$p'(z) = 15(z^2 + 1)(z - 1)(z - 3)$$

Mathematica

Here is a tiled diagram produced by Mathematica.

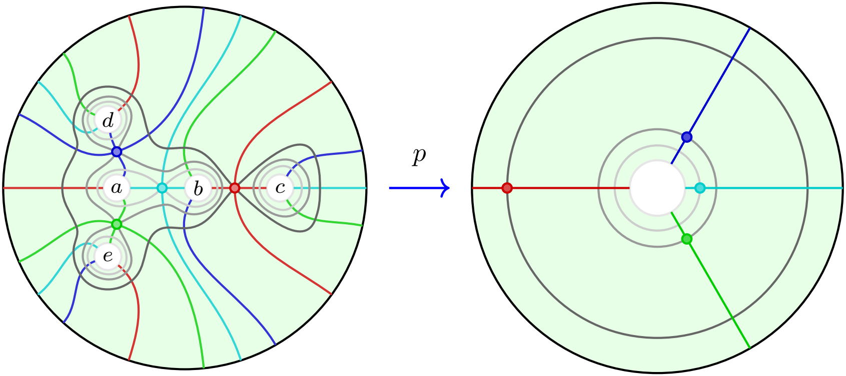


Technical Aside 1: Annular Metric



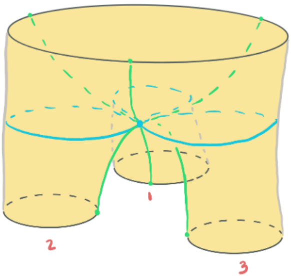
We give $\mathbb{C}_\mathbf{0}$ the bounded metric of the open cylinder $\mathbb{T} \times \mathbb{I}^{\text{int}}$ (via stereographic projection followed by radial projection).
Rays and circles in $\mathbb{C}_\mathbf{0}$ = longitudes and latitudes in $\mathbb{S}^2$ = horizontal circles and vertical lines in $\mathbb{T} \times \mathbb{I}$.

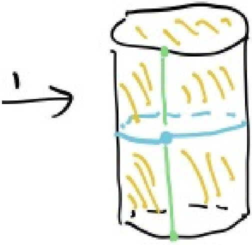
An Example: Square Tiling

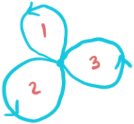


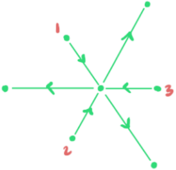
pull back both latitudes and longitudes
$\Longrightarrow$ square tiling of a disk with n holes

Technical Aside 2: Multipedal pants









Our Main Theorems

Theorem (Annulus)

$\textsc{Poly}_d(\mathbb{C}_{\mathbf{0}}) \cong \operatorname{int}(\textsc{BrAnn}_d)$.

Theorem (Rectangle)

$\textsc{Poly}_d(\mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cong \operatorname{int}(\textsc{BrRect}_d)$.

Theorem (Circle)

$\textsc{Poly}_d(\mathbb{T}) \cong \textsc{BrCirc}_d =$ *the dual braid complex.*

Theorem (Line)

$\textsc{Poly}_d(\mathbb{R}_{>0}) \cong$ *the "interior" of* $\textsc{BrLine}_d = |\textsc{NCPart}_d|$.

Thank You

References:

- Critical Points, Critical Values, and a Determinant Identity for Complex Polynomials (w/ M.Dougherty) *Proceedings AMS* 2020 (`arXiv:1908.10477`)
- Geometric Combinatorics of Polynomials I: The Case of a Single Polynomial (w/ M.Dougherty) *J.Algebra 2022* (`arXiv:2104.07609`)
- Dual Braids and the Braid Arrangement (Survey)
- Geometric Combinatorics of Polynomials II: Branched Rectangles and Basketballs (w/ M.Dougherty) - in prep.
- Geometric Combinatorics of Polynomials III: Branched Annuli and the Braid Arrangement (w/ M.Dougherty) - in prep.