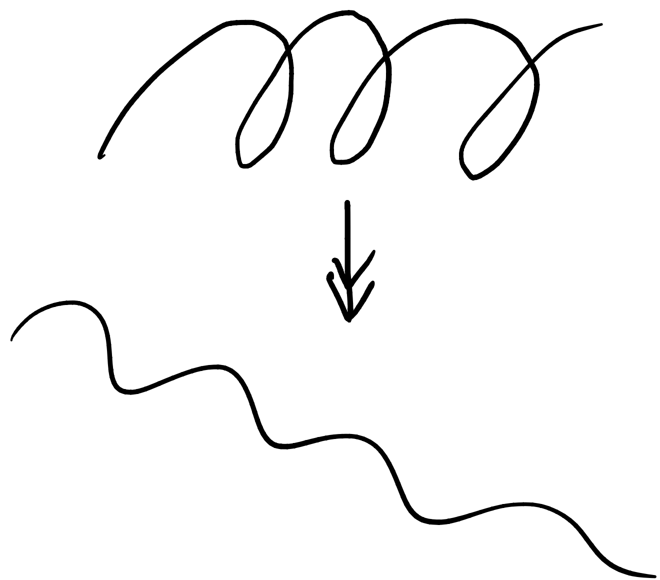
Einstein-Yang-Mills & Kaluza-Klein



J.T. Hartwig, Feb 2023

Sources:

- Hamilton, "Mathematical Gauge Theory"
- Lovelock, Rund, "Tensors, Differential Forms and Variational Principles"
- Coquereaux, Jadczyk, "Riemannian Geometry, Fiber Bundles, Kaluza-Klein Theories and all that"
- Mosel, "Fields, Symmetries, and Quarks"

Plan

1. Geometry
2. Algebra
3. Physics

Plan

1. Geometry (Connections on PFBs)
2. Algebra (Covariant differentiation)
3. Physics (Yang-Mills Lagrangian)

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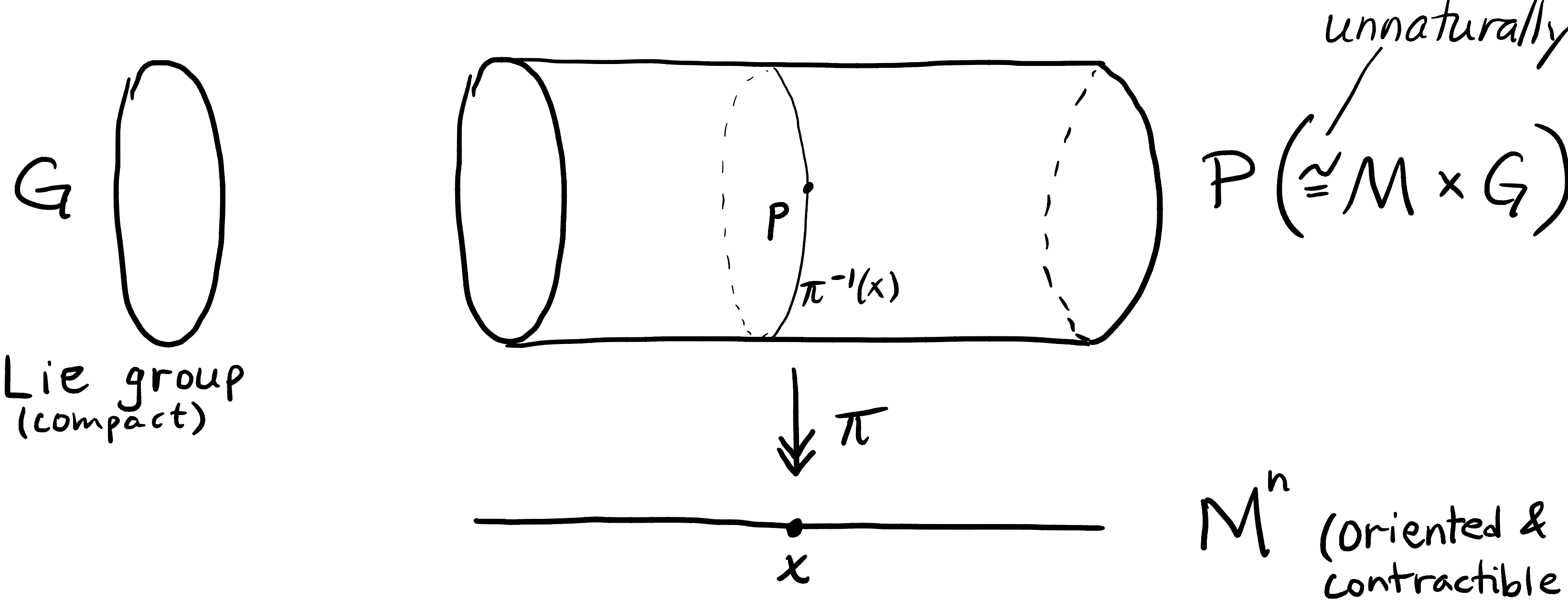
4. Gravity
5. Kaluza-Klein

$\left.\begin{array}{l} 4. \\ 5. \end{array}\right\}$ next time

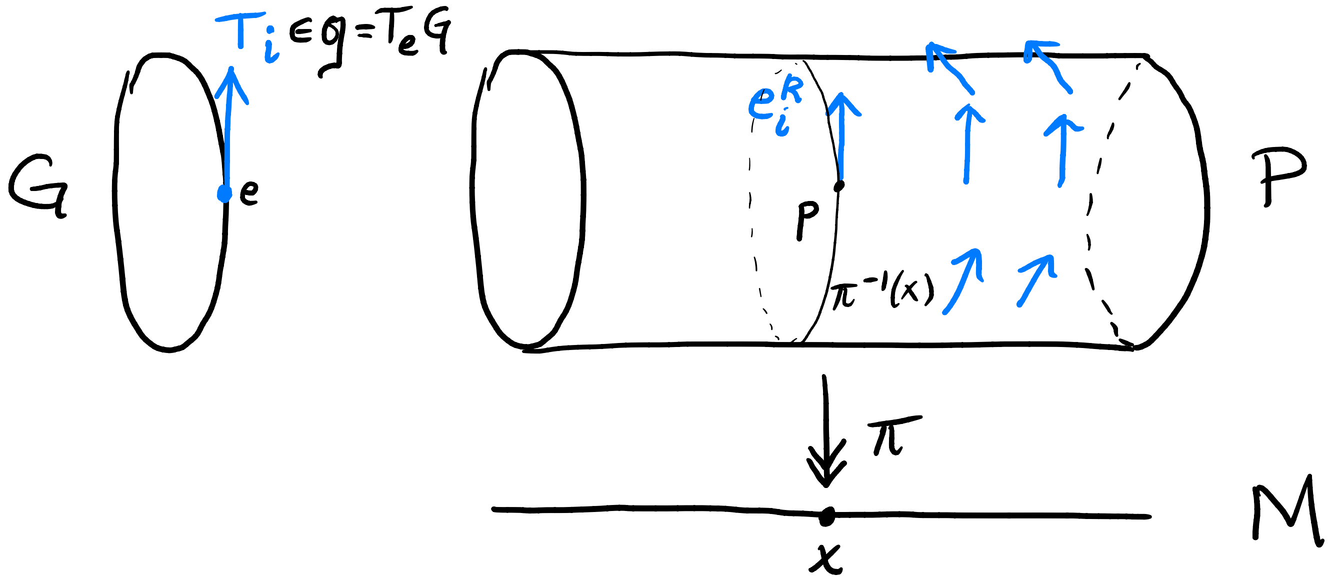
1. Anatomy of a principal fiber bundle.



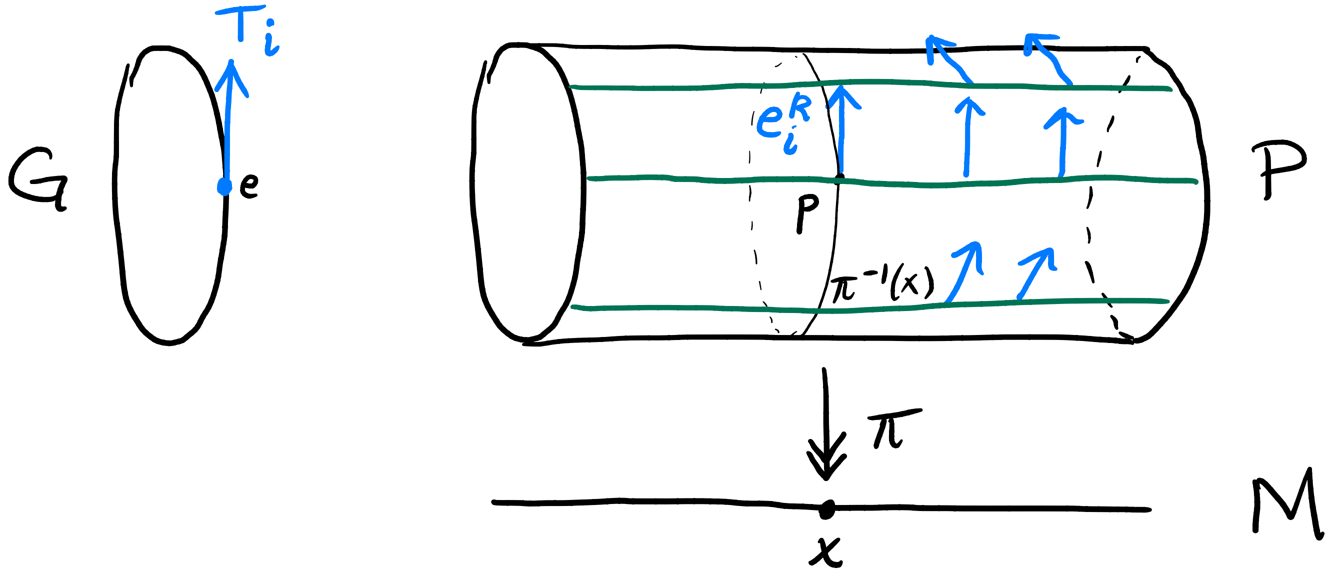
1) $P \circlearrowleft G$ free action

2) G-orbits = fibers of π

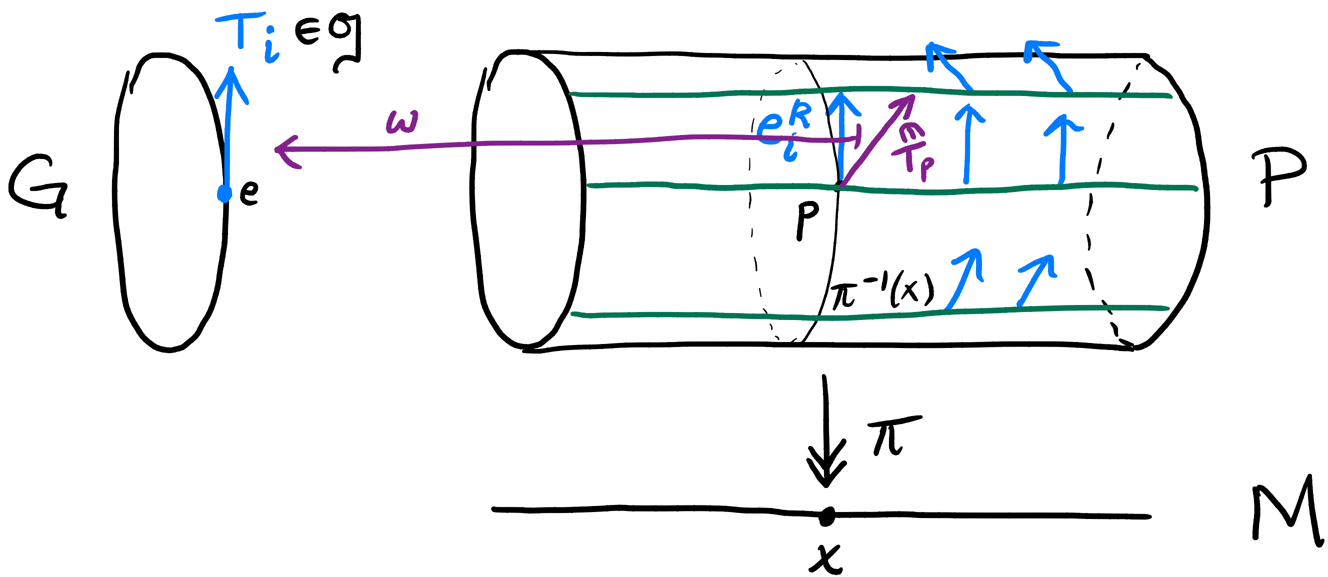
$$p.G = \pi^{-1}(\pi(p))$$



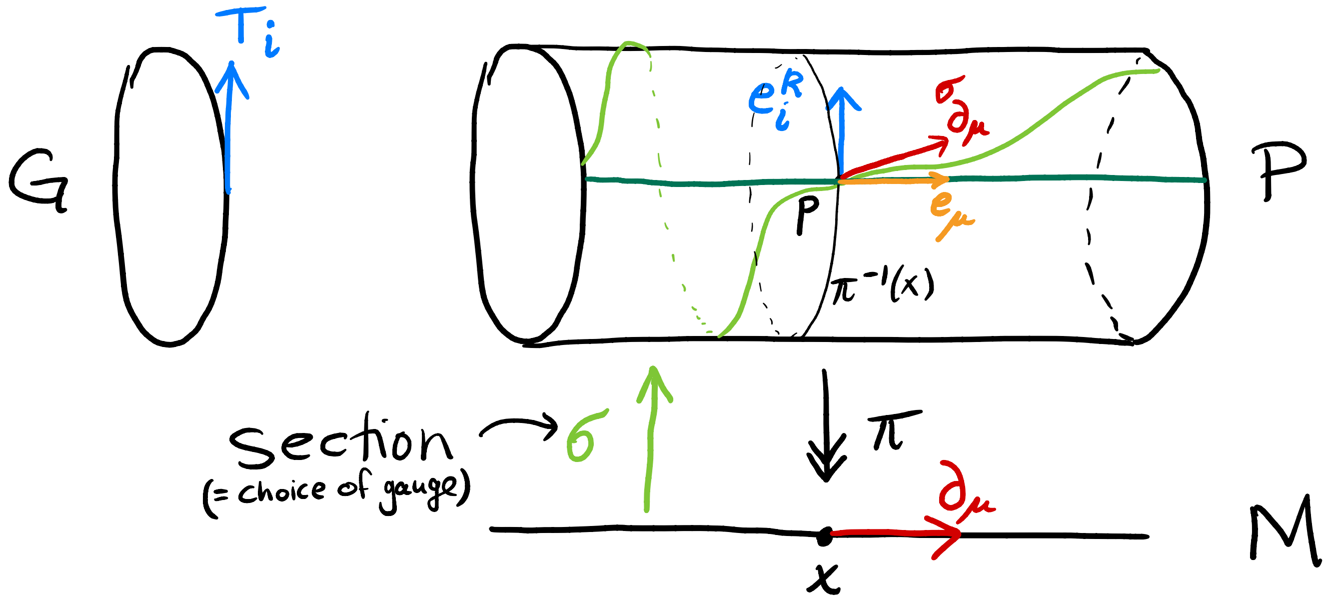
- To each $T_i \in \mathfrak{g}$ corresponds a fundamental vectorfield e_i^R on P: $\quad e_i^R(p) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[p \cdot \exp(\varepsilon T_i) \right]$
- The vertical subspace $V_p \subset T_p P$ is $\operatorname{span}\{e_i^R(p)\}_i$.



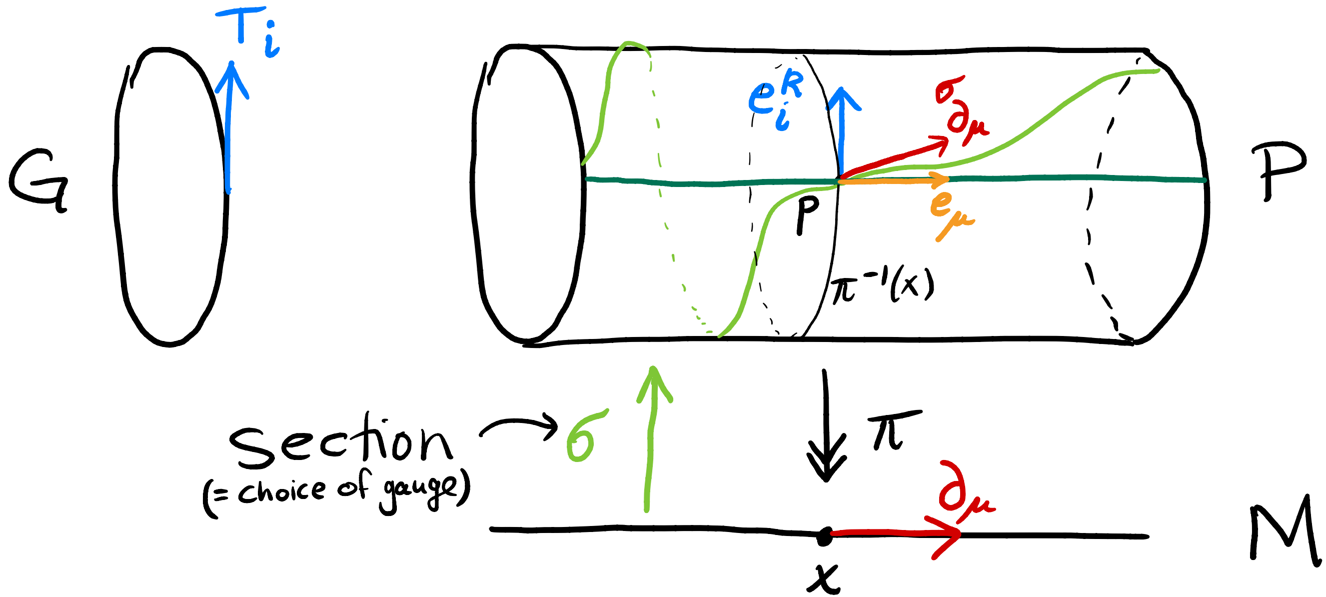
- An (Ehresmann) connection (H_p) is a distribution with
$$T_p = V_p \oplus H_p \quad \text{at every } p \in P$$
- H_p is the horizontal subspace



- Projection onto V_p along H_p followed by $V_p \cong \mathfrak{g}$ produces a connection form ω on P, which by definition is a $\mathfrak{g}$-valued 1-form with
  1) $r_{g*}\omega = g^{-1}\omega g$ and 2) $\omega(e_i^R) = T_i$
- Conversely, $H_p := \ker \omega_p$ defines (H_p) from ω.



- We can lift a tangent vector ∂_μ on M to a tangent vector e_μ on P by requiring $\omega(e_\mu)=0$.
- Given a section $\sigma : M \to P$ $\quad (\pi \circ \sigma = \mathrm{Id}_M)$ we get another lift: ${}^{\sigma}\partial_\mu = \sigma_*(\partial_\mu)$.
- Note: $\pi_*({}^{\sigma}\partial_\mu - e_\mu) = 0$ hence ${}^{\sigma}\partial_\mu - e_\mu \in V_p$.

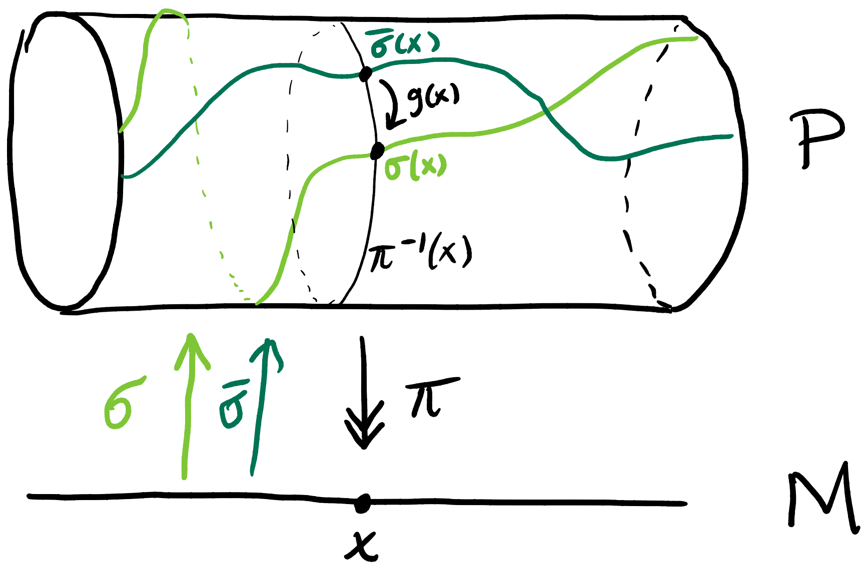


- Define ${}^\sigma\!A_\mu{}^i(x)$ by

$$e_\mu = {}^\sigma\partial_\mu - {}^\sigma\!A_\mu{}^i(x)\, e_i^R$$

- ${}^\sigma\!\mathcal{A}_\mu = {}^\sigma\!A_\mu{}^i\, T_i$ is the Yang-Mills gauge field (of ω).

If $\sigma, \bar{\sigma} : M \longrightarrow P$ are two sections: (choices of gauge)



then $\exists\, g : M \longrightarrow G$ such that

$$\sigma(x) = \bar{\sigma}(x) . g(x) \quad \forall x \in M.$$

Under a change of $\overset{\text{(section)}}{\text{gauge}}$ $\sigma \rightsquigarrow \bar{\sigma}$ the Yang-Mills field transforms as

(Mauer-Cartan form)

$$\boxed{{}^{\bar{\sigma}}\mathbb{A}_\mu = g\,{}^{\sigma}\mathbb{A}_\mu\, g^{-1} + \underbrace{g\,\partial_\mu g^{-1}}_{-(\partial_\mu \varepsilon^a(x))\,T_a \ \text{if}\ g(x) = \exp(\varepsilon^a(x) T_a)}}$$

2. Covariant Differentiation

Fix a representation

$$\rho : G \longrightarrow GL(V) \qquad g.v := \rho(g)v$$

and consider the associated vector bundle

$$[p,v] \in P \times_G V = P \times V \big/ (p.g, v) \sim (p, g.v)$$

$$\begin{array}{ccc} [p,v] & \in & P \times_G V \\ \big\downarrow & & \big\downarrow \\ \pi(p) & \in & M \end{array}$$

We want to <u>differentiate sections</u> $\varphi : M \longrightarrow P \times_G V$

Let $\varphi : M \to P \times_G V$ be a section.

For each section $\sigma : M \to P$, define ${}^{\sigma}\varphi : M \to V$ by:

$$\varphi(x) = [\sigma(x), {}^{\sigma}\varphi(x)]$$

What is the relationship between ${}^{\sigma}\varphi$, ${}^{\bar{\sigma}}\varphi$?

$$\varphi(x) = [\sigma(x), {}^{\sigma}\varphi(x)] = [\bar{\sigma}(x).g(x), {}^{\sigma}\varphi(x)] =$$

$$= [\bar{\sigma}(x), g(x).{}^{\sigma}\varphi(x)]$$

$$\boxed{{}^{\bar{\sigma}}\varphi(x) = g(x).{}^{\sigma}\varphi(x)}$$

Covariant Differentiation.

$$\partial_\mu {}^{\bar\sigma}\varphi = \partial_\mu\left(g(x).{}^\sigma\varphi\right) = \partial_\mu\Big[\underbrace{\exp(\varepsilon^i(x)\overset{\substack{\text{basis for } \mathfrak{g}=\mathrm{Lie}\,G\\ \downarrow}}{T_i})}_{g(x)}\ {}^\sigma\varphi\Big] =$$

$$= \underbrace{g(x)\,\partial_\mu {}^\sigma\varphi}_{\text{OK}} + \underbrace{g(x).\left(\partial_\mu \varepsilon^i(x)\right)T_i.{}^\sigma\varphi}_{\text{didn't want}}$$

"**Ansatz**": $$D_\mu {}^\sigma\varphi = \partial_\mu {}^\sigma\varphi + {}^\sigma A_\mu . {}^\sigma\varphi$$

$$\left({}^\sigma A_\mu = {}^\sigma A_\mu{}^i\, T_i\right)$$

Then

$$D_\mu {}^{\bar\sigma}\varphi = g(x)\, D_\mu {}^\sigma\varphi$$

$$\Longleftrightarrow \quad {}^{\bar\sigma}A_\mu(x) = g(x)\,{}^\sigma A_\mu(x)\, g(x)^{-1} - \left(\partial_\mu \varepsilon^i(x)\right)T_i \qquad !$$

Proof:

$$D_\mu\,{}^{\bar\sigma}\varphi = \underbrace{\partial_\mu\left(g\,{}^{\sigma}\varphi\right)} + \underbrace{{}^{\bar\sigma}A_\mu\; g}\,{}^{\sigma}\varphi \;=\; g\; D_\mu\,{}^{\sigma}\varphi$$

$$= \left(g\,\partial_\mu \cancel{\varepsilon^i T_i}\right){}^{\sigma}\varphi + g\,\partial_\mu{}^{\sigma}\varphi \qquad = g\left({}^{\sigma}A_\mu - \partial_\mu \cancel{\varepsilon^i T_i}\right)$$

So

$$\left\{D_\mu\,{}^{\sigma}\varphi\right\}_\sigma$$

<u>defines another section of the associated vector bundle.</u>

3. Field strength & Curvature (Drop σ from notation)

$$[D_\mu, D_\nu] = [\partial_\mu + \mathbb{A}_\mu, \partial_\nu + \mathbb{A}_\nu] =$$

$$= \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + [\mathbb{A}_\mu, \mathbb{A}_\nu]$$

Def $\mathbb{F}_{\mu\nu} = [D_\mu, D_\nu]$ is the field strength or curvature of the connection (YM field)

Ex. $G = U(1) = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$, $M = \mathbb{R}^{1,3}$

Then $\mathbb{F}_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{bmatrix}$

Yang-Mills Lagrangian

$(M, g_{\mu\nu})$ Riemannian mfd.

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\left(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right)\cdot\sqrt{|g|} \qquad \text{(scalar density)}$$

$$= -\frac{1}{4} g^{\alpha\mu} g^{\beta\nu}\,\mathrm{Tr}\,\mathbb{F}_{\alpha\beta}\,\mathbb{F}_{\mu\nu}\cdot\sqrt{|g|}$$

$$S = \int_M \mathcal{L}\, d^n x \qquad \text{(action)}$$

or some fixed $K \subset\subset M$

$\underline{G = U(1)}$:

compact support

$$0 = \delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int \mathcal{L}[A_\mu + \varepsilon \xi_\mu]\, d^n x =$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial A_\mu} \xi_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \partial_\nu \xi_\mu \right) d^n x$$

"integration by parts" (divergence thm)

$$= \int \left(\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) \xi_\mu \, d^n x \qquad \forall\, \xi_\mu$$

$$\Longleftrightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0 \qquad \text{(Euler-Lagrange)}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$$

With $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \quad \text{while}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}(F^{\mu\nu} - F^{\nu\mu}) \cdot 2 = F^{\nu\mu}$$

$$\left.\begin{array}{l} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \iff \boxed{F^{\nu\mu}{}_{,\mu} = 0} \\ \text{Automatic: } \boxed{F^{\mu\nu,\tau} + F^{\nu\tau,\mu} + F^{\tau\mu,\nu} = 0} \end{array}\right\} \text{Maxwell's Eqs}$$

$G = U(1) \times SU(2) \times SU(3)$ dim $1+3+8$

$\Rightarrow$ (unbroken) electroweak + strong gauge fields

$\mathbb{A}_\mu = A_\mu{}^i$

$i = 1$	photon	γ
$2, 3, 4$	$W^\pm$, Z^0	
$5, \ldots, 12$	gluons	g^a