Moduli of Representations of Clannish Algebras

Arxiv: 2208.00336

Cody Gilbert (U. Iowa)

I. Overview and Context

- Throughout, $k = \overline{k}$ and char $k = 0$.
- All quivers will be finite and connected.
- All algebras will be assumed to be associative and finite dimensional over k.

- Moduli of representations of finite dimensional algebras were introduced by King in [King '94]

- Moduli of representations can be arbitrarily complicated

[Hille '96, Huisgen-Zimmermann '98]

Conjecture [Carroll–Chindris '15]:
Let (Q, I) be a bound quiver, and $A = {}^{kQ}/_{I}$ its bound quiver algebra. If A is tame, then for any irreducible component $Z \subset \mathrm{rep}_Q(I, d)$ and any weight θ s.t $Z_\theta^{ss} \neq \emptyset$, $\mathcal{M}(Z)_\theta^{ss}$ is a product of projective spaces.

- The decomposition holds for the following classes of algebras:

  - Concealed–Canonical Algebras [Domokos–Lenzing '02]
  - Tame–tilted Algebras [Chindris '13]
  - Quasi–tilted Algebras [Bobinski '14]
  - Acyclic Gentle Algebras [Carroll–Chindris '15]
  - Special Biserial Algebras [Carroll–Chindris–Kinser–Weyman '20]

Main Theorem (Gilbert): Let $\Lambda = {}^{kQ}/_{I}$ be a clannish algebra (for example, a skewed-gentle algebra). Then any irreducible component of a moduli space $\mathcal{M}(\Lambda, d)_\theta^{ss}$ is isomorphic to a product of projective spaces.

- Clannish algebras were introduced in [Crawley-Boevey '89], and Skewed-gentle algebras in [Geiss-de la Peña '99].

- Applications for clans and clannish algebras have surfaced in Cluster theory [Qiu-Zhou '17, Amiot-Plamondon '21], meanwhile work involving skewed-gentle algebras includes [Chen-Lu '15 & '17, Amiot-Brüstle '19, He-Zhou-Zhu '20, Labardini-Fragoso-Schroll-Valdivieso '22].

II. Moduli of Representations of Algebras

- Throughout this section, $A = kQ/I$

- For a fixed $d \in \mathbb{N}^{Q_0}$, we define the representation variety

$$\mathrm{rep}_Q(I,d) := \left\{ M \in \prod_{a \in Q_1} \mathrm{Mat}_{d(ha) \times d(ta)}(k) \;\middle|\; M(r) = 0, \text{ for all } r \in I \right\}.$$

- With $GL(d) = \prod_{x \in Q_0} GL(d(x), k)$, we have an action $GL(d) \curvearrowright \mathrm{rep}_Q(I,d)$:

$$(\varphi \cdot M)(a) := \varphi(ha) \cdot M(a) \cdot \varphi^{-1}(ta), \text{ where } a \in Q_1,\ \varphi \in GL(d).$$

- An irreducible component $Z \subseteq \mathrm{rep}_Q(I,d)$ is said to be indecomposable (Schur) if its general points are indecomposable (Schur).

- For $Z \subseteq \mathrm{rep}_Q(I,d)$ an irreducible, closed, $GL(d)$-invariant subvariety and $\theta \in \mathbb{Z}^{Q_0}$,

(i) $Z^{ss}_\theta = \{ M \in Z \mid \theta(\dim M) = 0 \ \& \ \theta(\dim M') \leq 0 \text{ for } M' \leq M \}$

(ii) $Z^{s}_\theta = \{ M \in Z \mid \theta(\dim M) = 0 \ \& \ \theta(\dim M') < 0 \text{ for } 0 < M' < M \}$

- The category $\mathrm{rep}_Q(I)^{ss}_\theta$ of θ-semistable representations of A is Abelian with simple objects consisting of θ-stable representations.

<u>Definition:</u> For an irreducible, θ-semistable variety $Z \subseteq \mathrm{rep}_Q(I,d)$ we let

$$\mathcal{M}(Z)^{ss}_\theta := \mathrm{Proj}\left(\bigoplus_{n \geq 0} SI(Z)_{n\theta}\right)$$

denote the corresponding moduli space of Z, whose points are in bijection with the closed $GL(d)$-orbits in Z^{ss}_θ.

- From [CC15b], for A tame and $Z \subset \mathrm{rep}_Q(I,d)$ a θ-stable, irreducible component, if Z is normal, then $\mathcal{M}(Z)^{ss}_\theta$ is either a point or $\mathbb{P}^1$.

• The following theorem, combined with the above observation, allows one to conclude $\mathcal{M}(Z)^{ss}_\theta$ is a product of projective spaces, as long as we can prove $\tilde{Z}$ below is normal.

Theorem [Chindris-Kinser '18]:

For $Z \subseteq \operatorname{rep}_Q(I,d)^{ss}_\theta$ an irreducible Component,

$Z = m_1 Z_1 \dotplus \cdots \dotplus m_r Z_r$ a θ-stable decomposition and

$\tilde{Z} = \overline{Z_1^{\oplus m_1} \oplus \cdots \oplus Z_r^{\oplus m_r}}$, we have

(i) $\mathcal{M}(\tilde{Z})^{ss}_\theta = \mathcal{M}(Z)^{ss}_\theta$ whenever $\mathcal{M}(Z)^{ss}_\theta$ is irreducible.

(ii) If Z_1 is an orbit-closure, then

$$\mathcal{M}\left(\overline{Z_1^{\oplus m_1} \oplus \cdots \oplus Z_r^{\oplus m_r}}\right)^{ss}_\theta \simeq \mathcal{M}\left(\overline{Z_2^{\oplus m_2} \oplus \cdots \oplus Z_r^{\oplus m_r}}\right)^{ss}_\theta$$

(iii) There exists a finite, birational map

$$\Psi : S^{m_1}(\mathcal{M}(Z_1)^{ss}_\theta) \times \cdots \times S^{m_r}(\mathcal{M}(Z_r)^{ss}_\theta) \to \mathcal{M}(\tilde{Z})^{ss}_\theta$$

which is an isomorphism when $\mathcal{M}(\tilde{Z})^{ss}_\theta$ is normal.

III. Background on Tame Algebras

(i) Moduli Spaces of Tame Algebras

- $A = {}^{kQ}/I$ is a finite-dimensional tame algebra.

Theorem [CC 15b][Geiss-Labardini Fragoso-Schröer '22]:

Let $Z \subset \mathrm{rep}_Q(I,d)$ be an indecomposable, irreducible component.
Then $c_A(Z) := \min\{\dim(Z) - \dim \mathcal{O}_M \mid M \in Z\} \in \{0,1\}$.
Furthermore,

- $c_A(Z) = 0$ iff Z contains indecomposable M with $Z = \overline{\mathcal{O}_M}$.
- $c_A(Z) = 1$ iff Z contains a rational curve C such that the points of C are pairwise non-isomorphic indecomposables with $Z = \overline{\bigcup_{M \in C} \mathcal{O}_M}$.

Corollary:
If $Z \subseteq \mathrm{rep}_Q(I,d)$ is an irreducible component, then
$\dim Z \leq \dim GL(d)$.

Lemma: Let $A = {}^{kQ}/_{I}$ and $B = {}^{kQ}/_{I'}$ be f.d tame algebras w/ $I' \subset I$. Let $Z_i \subseteq \operatorname{rep}_Q(I, d_i)$, $1 \leq i \leq m$, be irreducible components satisfying:

- each Z_i is Schur;
- $c_A(Z_i) = 1$;
- $\operatorname{Hom}_A(M_i, M_j) = 0$ for $i \neq j$ and general $M_i \in Z_i$, $M_j \in Z_j$.

With $d = \sum_{i=1}^{m} d_i$, then $Z = \overline{Z_1 \oplus \cdots \oplus Z_m}$ is an irreducible component of $\operatorname{rep}_Q(I', d)$ w.r.t the closed embedding $\operatorname{rep}_Q(I, d) \subset \operatorname{rep}_Q(I', d)$.

(ii) Skewed-Gentle and Clannish Algebras

Definition: A gentle pair is a pair (Q, I) given by a quiver Q and an ideal I generated by paths of length two in Q such that

- for each $i \in Q_0$, there are at most two arrows with source i, and at most two arrows with target i;
- for each arrow $\alpha: i \to j$ in Q_1, there exists at most one arrow β with target i s.t $\beta\alpha \in I$ and at most one arrow β' w/ target i s.t $\beta'\alpha \notin I$;
- for each arrow $\alpha: i \to j$ in Q_1, there exists at most one arrow β with source j s.t $\alpha\beta \in I$ and at most one arrow β' w/ source i s.t $\alpha\beta \notin I$;

- the algebra $A = kQ/I$ is finite dimensional.

- With Q a quiver, we let $Q_1^{sp} \subset Q_1$ be a subset of loops of Q_1. Elements of Q_1^{sp} are called special loops.

- When defining a set R of relations on Q, we always include the set of relations:

$$R^{sp} = \{e^2 - e \mid e \in Q_1^{sp}\}$$

So $R = R^{sp} \cup \mathcal{I}$ where $\mathcal{I}$ is a set of zero-relations.

Definition: An algebra $A = kQ/(I + \langle R^{sp} \rangle)$ is called skewed-gentle if $(Q, I + \langle e^2, e \in Q_1^{sp} \rangle)$ is a gentle pair, where I is an ideal generated by paths of length two.

Definition: With $R = R^{sp} \cup \mathcal{I}$ and $I = \langle R \rangle$, the algebra $\Lambda = kQ/I$ is clannish when the following hold:

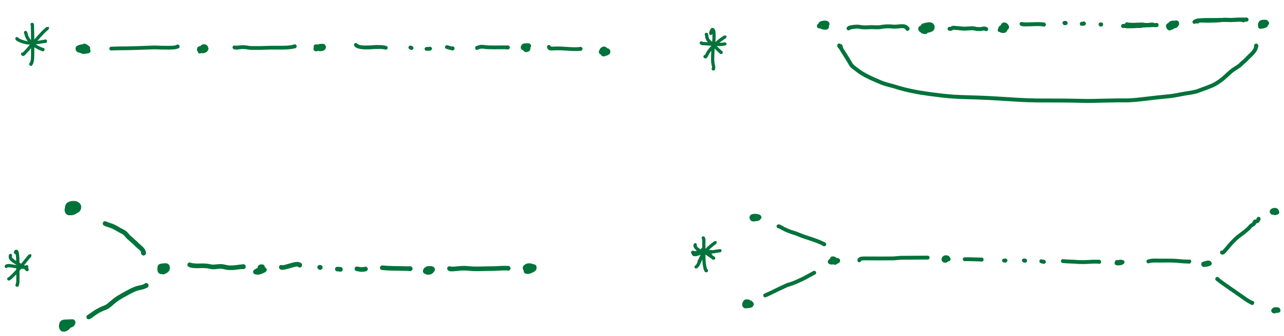
(C1) None of the relations in $\mathcal{I}$ begin or end with a special loop.

(C2) For each vertex $v \in Q_0$, there are at most two arrows with head v and at most two arrows with tail v.

(C3) For any arrow $b \in Q_1 \setminus Q_1^{sp}$ there is at most one arrow $a \in Q_1$ with $ba \notin I$ and at most one arrow $c \in Q_1$ with $cb \notin I$. Note: $a, c \in Q_1$ can be ordinary or special.

- Clannish algebras generalize special biserial algebras in that all but finitely many indecomposable representations of a clannish algebra are determined by walks of the following forms:

<u>Strings:</u> <u>Bands:</u>



<u>Observation:</u> With $Q := 1 \xrightarrow{a} 2 \circlearrowleft^{e}$ and $I = \langle e^2 - e \rangle$, kQ/I is isomorphic to the path algebra of $Q' := 1 \rightrightarrows \{2^+, 2^-\}$ With $k\langle e \rangle / \langle e^2 - e \rangle \simeq \underbrace{ke}_{2^+} \times \underbrace{k(1-e)}_{2^-}$.

- Extending the above method gives a way in which one can construct bound quiver algebras isomorphic to clannish algebras. More detail can be found in [Amiot–Brüstle '19].

<u>Example:</u> Consider $\Lambda = {}^{kQ}/I$ where Q is the quiver given by

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[loop above, "f"] \arrow[r, "b"] & 3 \arrow[r, "c"] & 4 \end{tikzcd}$$

and $I = \langle ba, cbfa, f^2 - f \rangle$. The algebra Λ is isomorphic

to the algebra $\Lambda' = {}^{kQ'}/I'$ given by

$$Q' := \begin{tikzcd} & 2^+ \arrow[dr, "b^+"] & & \\ 1 \arrow[ur, "a^+"] \arrow[dr, "a^-"'] & & 3 \arrow[r, "c"] & 4 \\ & 2^- \arrow[ur, "b^-"'] & & \end{tikzcd}$$

and relations $I' = \langle b^+a^+ + b^-a^-, c\, b^+a^+ \rangle$.

- As cb^+ and cb^- are nontrivial, this algebra is not a quotient of a gentle algebra.

IV. Proof of Main Theorem

- Let $\Lambda = {}^{kQ}/_{I}$ be a clannish algebra.

Lemma: There exists an ideal $J \subseteq I \subset kQ$ such that $\Lambda' := {}^{kQ}/_{J}$ is a skewed-gentle algebra. As such, any clannish algebra Λ is a quotient of a skewed-gentle algebra Λ'.

Proposition: Let $\Lambda' = {}^{kQ}/_{J}$ be a skewed-gentle algebra and d be a dimension vector. If $Z \subset \mathrm{rep}_Q(J, d)$ is an irreducible component, then Z is normal.

Proof Idea: One can decompose Z as a product of varieties

$$Z \cong \prod_{i=1}^{L} Z_i' \times \prod_{k=L+1}^{t} Z_k''$$

where the Z_i' are varieties of idempotent matrices and the Z_k'' are irreducible components of representation varieties of gentle algebras. As such, Z is a product of normal varieties.

Lemma: Let $\Lambda = kQ/I$ and $\Lambda' = kQ/J$ be as above. Let $Z_i \subseteq \mathrm{rep}_Q(I, d_i)$, $1 \le i \le m$, be irreducible components satisfying:

- each Z_i is Schur;
- $c_A(Z_i) = 1$;
- $\mathrm{Hom}_A(M_i, M_j) = 0$ for $i \neq j$ and general $M_i \in Z_i$, $M_j \in Z_j$.

With $d = \sum_{i=1}^{m} d_i$, then $Z = \overline{Z_1 \oplus \cdots \oplus Z_m}$ is an irreducible component of $\mathrm{rep}_Q(J, d)$ w.r.t the closed embedding $\mathrm{rep}_Q(I, d) \subset \mathrm{rep}_Q(J, d)$.

As such, Z is normal.

Theorem:
Let Λ be clannish. Then any irreducible component of a moduli space $\mathcal{M}(\Lambda, d)^{ss}_{\theta}$ is isomorphic to a product of projective spaces.

Proof idea: If Y is an irreducible component of $\mathcal{M}(\Lambda, d)^{ss}_{\theta}$, then there exists $Z \subset \mathrm{rep}_Q(I, d)$ with $Y = \mathcal{M}(Z)^{ss}_{\theta}$.

- We may write $Z = \overline{Z_1^{\oplus m_1} \oplus \cdots \oplus Z_r^{m_r}}$, where the Z_i are θ-stable.

Further, we may assume none of the Z_i are orbit closures.

- We have $\hom(Z_i, Z_j) = 0$ for all $1 \leq i, j \leq m$.

- By the lemma above, Z is normal. As such, $\mathcal{M}(Z)^{ss}_{\theta}$ is normal too.

- By moduli decomposition theorem,

$$\mathcal{M}(Z)^{ss}_{\theta} \cong \prod_{i=1}^{r} S^{m_i}(\mathcal{M}(Z_i)^{ss}_{\theta}) \cong \prod_{i=1}^{r} \mathbb{P}^{m_i}.$$

V. A Future Direction

- With kQ/I tame acyclic and

$$P_A(d) := \{ M \in \operatorname{rep}_Q(I, d) \mid \operatorname{pdim}_A M \leq 1 \}$$

one has that $C_A(d) = \overline{P_A(d)}$ is an irreducible component. The following problem was posed by Calin Chindris.

Problem: Let A be acyclic and tame. If $d \in \mathbb{N}^{Q_0}$ is such that $P_A(d) \neq \emptyset$, describe $\mathcal{M}(C_A(d))^{ss}_{\langle d, - \rangle}$.