Lattices.

Def. $V = (V, \langle\, ;\rangle)$

= finite dim $\mathbb{R}$ vector space with symm. bilinear form $\langle\,,\rangle$

Ex: $\mathbb{R}^{p,q}$ = $(p+q)$ dim $\mathbb{R}$-vector space with.

$\langle x, y\rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$

Def. $v^2 := \langle v | v\rangle$ called the norm of v.

Let $M \subseteq V$ be a $\mathbb{Z}$-lattice in V, ie.

M is the $\mathbb{Z}$-span of a basis of V and $\langle M, M\rangle \subseteq \mathbb{Q}$.

$M(k) := \{ v \in M : v^2 = k \}$.

$M^\vee := \{ v \in V : \langle v, M\rangle \subseteq \mathbb{Z} \}$. called the dual lattice.

M is integral if $\langle M, M\rangle \in \mathbb{Z}$ ie. $M \subseteq M^\vee$.

M is self dual if $M = M^\vee$.

M is even if $v^2 \in 2\mathbb{Z}$ $\forall\, v \in M$.

Ex: Even lattices are integral.

Ex: Let M be an integral lattice in $\mathbb{R}^n$. Then

$|M^\vee / M| = \mathrm{Vol}(V/M)^2$.

If M is an even lattice, we'll call $M(2)$, the set of roots of M.

Some important lattices

Ex: $\mathbb{Z}^{p,q}$, $II^{p,q}$: $\subseteq \mathbb{R}^{p,q}$, $I^{p,q} = \{x : x_i \in \mathbb{Z}$ for all i

$$II_{p,q} = \{ x : 2x_i, (x_i - x_j) \in \mathbb{Z} \text{ for all } i,j,\ \sum_i' x_i \in 2\mathbb{Z} \}.$$

Def A (simply laced) root lattice is a + definite even lattice spanned by its roots.

Ex: (of root lattices):

h

$A_n = \{ x \in \mathbb{Z}^{n+1} : \Sigma' x_i = 0 \} \subseteq V = \{ x \in \mathbb{R}^{n+1} : \Sigma' x_i = 0 \}$, $n \geq 1$ $n+1$

$D_n = \{ x \in \mathbb{Z}^n : \Sigma' x_i \in 2\mathbb{Z} \} \subseteq \mathbb{R}^n$, $n \geq 4$ $2n-2$

$E_8 = II_{8,0} \subseteq \mathbb{R}^8$ 30

$E_7 = \perp$ complement of any root in E_8. 18

$E_6 = \perp$ ...... A_2 in E_8. 12

Thm 1. The "irreducible" root lattices are A_n, D_n, E_6, E_7, E_8.

All root lattices are direct sums of these.

Def. The Coxeter number of an irreducible root lattice Λ is $h = \dfrac{|\Lambda(2)|}{rk(\Lambda)}$.

Thm 2. Let M be a self dual lattice of signature (m,n):

(a) If M is in indefinite and odd, then $M \simeq I_{m,n}$.

(b) If M is even, then $m - n \equiv 0 \bmod 8$.

(c) If M is indefinite and even, then $M \simeq II_{m,n}$.

proof: See [Serre].

Remark. By thm 2.b. + definite even dual lattices of dim n.

can exist only for $n = 8, 16, 24, 32, \cdots$

n	+ definite even dual lattices of dim n.
8	There is just one, $II_{8,0} = E_8$.
16	There are two, $E_8 \perp E_8$, $II_{16,0}$
* 24	There are 24, of these 23 contain a root lattice of finite index, and the 24th is the Leech lattice which has no roots
32	There are more than 80 million. These have not been classified.

· Modular forms. Write $e(z) = \exp(2\pi i z)$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z})$; $H^2 = \{x+iy \in \mathbb{C} : y > 0\}$

Define $g.\tau = \dfrac{a\tau+b}{c\tau+d}$, and $j(g,\tau) = (c\tau+d)$

Verify. $(g,\tau) \mapsto g.\tau$ is a group action, and $j(gh,\tau) = j(g,h\tau)\, j(h,\tau)$

Given $f : H^2 \to \mathbb{C}$, define $f\big|_{2k}^{g}(\tau) = (c\tau+d)^{-2k} \cdot f(g\tau)$.

Verify that $(g,f) \mapsto f\big|_{2k}^{g}$ is a group action.

Def. A function $f : H^2 \to \mathbb{C}$ is a modular function/modular form of wt $2k$ if

(1) $(c\tau+d)^{-2k} f(g\tau) = f\big|_{2k}^{g}(\tau) = f(\tau)$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in H$

(2) " f is meromorphic/holomorphic on $H^2 \cup \{i\infty\}$. "

Let us explain the meaning of the 2^{nd} condition.

Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\Gamma = \langle S, T \rangle$.

Note: $S.z = -\frac{1}{z}$, $T.z = (z+1)$. So f satisfies condition (1)

if and only if $f(z+1) = f(z)$ and $f(-1/z) = z^{2k} f(z)$.

Let $f : H^2 \to \mathbb{C}$ is a meromorphic function such that

$f(z+1) = f(z)$. Then f induces a function $\tilde{f} : D^\times \to \mathbb{C}$

such that

$$\begin{tikzcd} H^2 \arrow[rr, "e(\cdot)"] \arrow[dr, "f"'] & & D^\times \arrow[dl, "\tilde{f}"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes.

Write $q = e(z)$ Then $\tilde{f}(q) = f(z)$.

[illegible]

Note $e(it) \to 0$ as $t \to \infty$, so we write $e(i\infty) = 0$.

Say f is meromorphic (resp) holomorphic at $z = i\infty$ if $\tilde{f}$ extends to a meromorphic (resp.) holomorphic function at $q = 0$.

Note that if f is meromorphic at $i\infty$, then it has a Laurent seri[es]

$$f(z) = \tilde{f}(q) = \sum_{n \gg -\infty}' a_n q^n = \sum_n' a_n \, e(nz).$$

If f is holomorphic at $i\infty$, we write $f(i\infty) = \tilde{f}(0) = a_0$.

Let M_{2k} = space of modular forms of wt $2k$.

$$M_{2k}^{0} = \{ f \in M_{2k} : f(i\infty) = 0 \}.$$

Examples of modular forms.

Let $\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}. = \{ g \in \Gamma : g\infty = \infty \}$

Then verify that $\Gamma_\infty \backslash \Gamma = \left\{ \begin{bmatrix} * & * \\ c & d \end{bmatrix} : \gcd(c,d) = 1 \right\}.$

Define $E_{2k}(\tau) = \sum'_{g \in \Gamma_\infty \backslash \Gamma} 1 \Big|_{2k}^{g} = \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-2k} = \frac{1}{2\zeta(2k)} \sum'_{(m,n) \neq (0,0)} (m\tau + n)^{-2k}$

where $k = 2, 3, \ldots$ and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

It is easy to verify that $E_{2k} \in \mathrm{Mod}_{2k}$ and

direct computation yields that E_{2k} has the Fourier series

$$E_{2k}(\tau) = 1 - \frac{4k}{b_{2k}} \cdot \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

where $\sigma_r(n) = \sum_{d|n} d^r.$ and b_{2k} are the Bernoulli #'s.

defined by $\dfrac{t}{e^t - 1} = 1 - \dfrac{t}{2} + \sum_{k=1}^{\infty} b_{2k} \dfrac{t^{2k}}{2k!}$

In particular $E_4 = (1 + 240q + \cdots) \in M_4$

$$E_6 = (1 - 504q + \cdots) \in M_6$$

So: $\Delta = (E_4^3 - E_6^2)/1728 = q + \cdots \in M_{12}^0.$

Theorem $\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$

· Basics of modular forms.

Let $p \in (\Gamma \backslash H^2)$ = the standard fundamental domain of Γ.

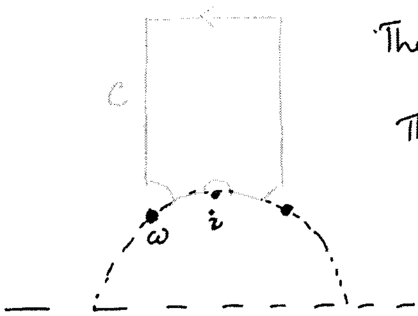
Let $\nu_p(f)$ = order of vanishing of f at p. $(\nu_\infty(f) = \nu_0(\tilde{f}))$

Thm. Let $f \neq 0$ be a modular function of wt $2k$. Then

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{\substack{p \in (\Gamma \backslash H) \\ p \neq i, \omega}}' \nu_p(f) = \frac{k}{6}. \quad -(*)$$

proof: Assume f has no zero on the contour shown.

Let $\alpha = \frac{1}{2\pi i} \frac{f'}{f} dz$. Then $\int_{\mathcal{C}} \alpha = \sum_{p \text{ inside } C} \nu_p(f)$ by residue theorem



The integral on the top horizonal segment give $-\nu_\infty(f)$

The integrals on the little arcs around i, ω, $-\bar{\omega}$

give $-\left(\frac{\nu_i(f)}{2}\right)$, $-\left(\frac{\nu_\omega(f)}{6}\right)$, $-\left(\frac{\nu_{-\bar{\omega}}(f)}{6}\right)$

the integrals on the vertical segments cancel out. and

$$f(Sz) = z^{2k} f(z) \Rightarrow \frac{d f(Sz)}{f(Sz)} = \frac{2k\, dz}{z} + \frac{df(z)}{f(z)}$$

$$\text{So } \int_\omega^i \alpha + \int_i^{-\bar{\omega}} \alpha = \frac{1}{2\pi i} \int_\omega^i \frac{df(z)}{f(z)} - \frac{df(Sz)}{f(Sz)} = \frac{1}{2\pi i} \int_\omega^i -2k \frac{dz}{z} = \frac{k}{6}. \quad \square$$

Cor. (1) $M_0 = \mathbb{C}\cdot 1$, $M_0^\circ = 0$, M_2

(2) For $k = 4, 6, 8, 10$, $M_k = \mathbb{C} E_k$. $M_k^\circ = 0$.

(3) For $2k \geq 12$, $M_{2k-12} \xrightarrow[\simeq]{\Delta\cdot} M_{2k}^\circ$

For example $M_{12} = \mathbb{C} E_{12} + \mathbb{C}\Delta$ is 2dimensional. .....

pf: For $2k<0$, $2k=2$, no way to satisfy (*). So $M_{2k}=0$

For $E_4 \in M_4$, Only way to satisfy (*) is $v_\omega(E_4)=1$, $v_p(E_4)=0$ o.w.

For $E_6 \in M_6$, ... - - is. $v_i(E_6)=1$, $v_p(E_6)=0$ ow.

$\Rightarrow$ $\Delta \neq 0$. Also $v_\infty(\Delta)=1$, So $(*) \Rightarrow$ $v_p(\Delta)=0$ $\forall p \neq \infty$.

ie Δ has a simple zero at $i\infty$. and no other zero on H^2.

So $f \in M_k^0 \Rightarrow \frac{1}{\Delta} f \in M_{k-12}$.

Since $M_{2k}=0$ for $k<0$, get $M_k^0=0$ for $k=4,6,8,10$.

So $\dim M_k \leq 1$, and $M_k = \mathbb{C} E_k$.

Cor. $\dim M_{2k} = \begin{cases} \lfloor k/6 \rfloor & \text{if } k \equiv 1 \bmod 6 \\ \lfloor k/6 \rfloor + 1 & \text{o.w.} \end{cases}$

Cor. $\bigoplus_k M_{2k} = \mathbb{C}[E_4, E_6]$.

In particular. $M_{14}^0 \simeq M_2 = 0$.

. Poisson Summation formula.

V = f. dim. $\mathbb{R}$ vector space. with + def. inner product $\langle\, ,\, \rangle$.

dx = normalized measure, vol(cube formed by orthonormal basis) = 1.

$M \subseteq V$, integral. lattice. ; $f : V \to \mathbb{C}$ Schwarz function.

ie $f \in C^\infty(V, \mathbb{C})$, and. $f(x)\, p(x) \longrightarrow 0$ for all polynomial p

Define $\hat{f}(u) = \int_V f(x) . e(-\langle x, u\rangle) dx$ Then

$$\sqrt{|M^\vee/M|} \sum_{v \in M}' f(v) = \sum_{\lambda \in M^\vee}' \hat{f}(\lambda).$$

proof. Define the M- periodic function $F(y) = \sum_{x \in M} f(y+x)$.

So $F : V/M \to \mathbb{C}$. For each $u \in M^\vee$, define

$$\chi_u : V/M \longrightarrow \mathbb{C}^\times \quad \text{by} \quad \chi_u(y) = e(\langle u, y\rangle).$$

$\{\chi_u : u \in M^\vee\}$ is a basis for the space of smooth function on the torus V/M., orthonormal w.r.t the hermitian form

$$(f, g) = \mathrm{vol}(V/M)^{-1} \int_{V/M} f(y)\, \overline{g(y)}\, dy$$

We have Fourier series $F(y) = \sum_{u \in M^\vee}' \hat{F}_u . \chi_u(y)$

where $\hat{F}_u = (F, \overline{\chi}_u)$

$$\mathrm{vol}(V/M)\, \hat{F}_u = \int_{V/M} \Sigma_{x \in M}\, f(y+x)\, \chi_{-u}(y)\, dy$$
$$= \int_V f(y)\, e(-\langle u, y\rangle) dy = \hat{f}(u).$$

So $\sum_{x \in \Lambda}' f(x) = F(0) = \sum_{u \in M^\vee}' \hat{F}_u = \mathrm{vol}(V/M)^{-1} \sum_{u \in M^\vee}' \hat{f}(u)$ $\square$

· Theta functions.

Def. Let Λ be an even lattice of rk n in $V=(V,\langle\ \rangle)$.

Define $\theta_\Lambda : H^2 \to \mathbb{C}$ by

$$\theta_\Lambda(\tau) = \sum_{x\in\Lambda}' e\left(\frac{x^2\tau}{2}\right) = \sum_{n=0}^{\infty} |\Lambda(2n)| \cdot q^n.$$

Thm [θ1] If Λ is an even self dual lattice of rk n, then $\theta_\Lambda \in M_{n/2}$

proof: We need to only show $\theta_\Lambda(-\frac{1}{\tau}) = \tau^{n/2} \cdot \theta_\Lambda(\tau)$. $\quad -(*)$

For $t\in(0,\infty)$, define $f: V \to \mathbb{C}$ by $f_t(x) = e^{-\pi x^2 t}$. Then $\theta_\Lambda(it) = \sum_{x\in\Lambda}' f_t(x)$

To compute $\hat{f}$, choose an orthonormal basis to identify $(V,\langle\rangle) \simeq \mathbb{R}^n$.

In this coord system $x^2 = \sum' x_j^2$ and $dx = \prod dx_j$. So

$f_t(x) = \prod_j \phi_t(x_j)$, where $\phi_t : \mathbb{R}\to\mathbb{R}$ is the function $\phi_t(x) = e^{-\pi x^2 t}$

One has $\hat{\phi}_t(u) = t^{-1/2} e^{-\pi u^2/t} = t^{-1/2} \cdot \phi_{1/t}(u)$. So

$\hat{f}_t(x) = \prod \hat{\phi}_t(x_j) = \prod t^{-1/2} \phi_{1/t}(u_j) = t^{-n/2} f_{1/t}(u)$.

Since $\Lambda^\vee = \Lambda$, Poisson summation gives

$$\theta(it) = \sum_{u\in\Lambda}' \hat{f}_t(u) = \sum_{u\in\Lambda} t^{-n/2} f_{1/t}(u) = t^{-n/2}\,\theta(i/t).$$

Since $n/2$ is even, $\theta(-\frac{1}{it}) = (it)^{n/2} \cdot \theta(it)$, so the two sides of $(*)$ agree on $\{it : t\in(0,\infty)\}$. Since the two sides are holomorphic $(*)$ must hold $\square$

• More on theta functions.

• Ex: Suppose Λ is a even self dual lattice of rank 8.

Then $\theta_\Lambda(\tau) = 1 + O(q) \in M_8 = \mathbb{C}\cdot E_4(\tau)$.

$\Rightarrow \quad \theta_\Lambda(\tau) = E_4(\tau) = 1 + 240q + \cdots$, !

So $\Lambda(2)$ is a rank 8 root system of size 240.

This easily implies $\Lambda(2) \cong E_8(2)$; So $\Lambda \simeq E_8$. !

The eq $\theta_{E_8}(\tau) = E_4(\tau)$, gives us a formula for # of lattice vectors of any length in E_8.

$$\#\left\{x \in \mathbb{Z}^8 \cup \left(\tfrac{1}{2}+\mathbb{Z}\right)^8 : \sum x_i = 0 \bmod 2, \sum_{i=1}^{8} x_i^2 = 2n\right\} = 240\,\sigma_3(n).$$

• For our application to 24^d lattices, we need a more general kind of theta function.

Def: For a polynomial $P : V \to \mathbb{C}$, write.

$$\theta_{\Lambda, P}(\tau) = \sum_{x \in \Lambda} P(x)\, e\left(\frac{x^2\tau}{2}\right).$$

Thm [θ2] Let Λ be an even self dual lattice of rk n. Choose $\alpha \in V$

Define $P_\alpha(x) = \langle x, \alpha\rangle^2 - \frac{1}{n}\alpha^2 \cdot x^2$. Then $\theta_{\Lambda, P_\alpha}(x) \in M^0_{\frac{n}{2}+2}$

proof: similar to proof of [θ1] again using Poission. Summation.

Only need Fourier transforms of $x\,\hat{\phi}_t(x)$ and $x\,\phi_t(x)$. $\square$

<u>Remark</u>: Thm [θ2] is special case of the following:

If $P : V \to \mathbb{C}$ is a spherical polynomial of deg. $\nu, > 0$

(ie. $\sum_j' \frac{\partial^2}{\partial x_j^2} P = 0$), then $\theta_{\Delta, P}(x) \in \underline{M}^0_{\frac{n}{2}+\nu}$.

Indeed, verify that P_α in [θ2] is a spherical poly of deg 2.

· Even self dual lattices of rk 24.

(Ref:: Venkov's paper in Conway and Sloane's book).

From here on, let $\Lambda \subseteq \mathbb{R}^{24}$ be an even self dual lattice of rk 24.

Lemma: Let $\alpha \in \mathbb{R}^{24}$. Then $\sum'_{y \in \Lambda(2)} (y, \alpha)^2 = \frac{1}{12} \alpha^2 \cdot |\Lambda(2)|$.

proof: We have $\theta_{\Lambda, P_\alpha}(\tau) = \sum_{x \in \Lambda} \left((x,\alpha)^2 - \frac{1}{24} \alpha^2 \cdot x^2 \right) q^{x^2/2} \in M^0_{14} \simeq M_2 = 0$

Equating coeff of q^n, get $\sum'_{x \in \Lambda(2n)} (x,\alpha)^2 - \frac{1}{24} \alpha^2 \cdot 2n |\Lambda(2n)| = 0$ □

Cor 1. Either $\Lambda(2) = \phi$ or $\mathrm{rk}(\Lambda(2)) = 24$.

proof: If $\mathrm{rk}(\Lambda(2)) < 24$, then choose $\alpha \in \mathbb{R}^{24} - \{0\}$, such that $\alpha \perp \Lambda(2)$, and then Lemma $\Rightarrow |\Lambda(2)| = 0$. □

Cor 2. All irreducible components of the root system $\Lambda(2)$ have the same coxeter number $h = |\Lambda(2)|/24$.

proof:

Fact: Let M be an irreducible root lattice, and $\alpha \in M(2)$.

Then $\# \{ v \in M(2) : \langle v, \alpha \rangle \neq 0 \} = 4h(M) - 6$.

Let $\beta(M) = \# \{ v \in M(2) : \langle v, \alpha \rangle = 1 \}$.

Note that the #'s above do not depend on α since the reflection group of M acts transitively on $M(2)$.

If $\langle v, \alpha \rangle \neq 0$, then $\langle v, \alpha \rangle = \pm 1$ or $\langle v, \alpha \rangle = \pm 2$

and in the second case $v = \pm\alpha$.

So $4h(M) - 6 = 2 + 2\beta(M)$, ie. $\beta(M) = 2h(M) - 4$.

Now take $\alpha \in \Lambda(2)$ and let M_α be the irreducible root sublattice of Λ containing α. Then Lemma $\Rightarrow$

$$\frac{1}{6}|\Lambda(2)| = \sum_{y \in \Lambda(2)}' \langle y, \alpha\rangle^2 = \sum_{\substack{y \in M_\alpha(2) \\ (y,\alpha) \neq 0}} \langle y, \alpha\rangle^2 = 2 \cdot 2^2 + 2 \cdot \beta(M_\alpha) = 4h(M_\alpha). \quad \Box$$

· root systems of 24 dim even self dual lattices

Thm Let Λ be an even self dual lattice of rk 24. with $\Lambda(2) \neq$

Then rk $\Lambda(2) = 24$. an each irreducible component of $\Lambda(2)$ has the same coxeter # $h = |\Lambda(2)|/24$.

Cor. The possible rk 24 root systems of Λ are.

ϕ, A_1^{24}, A_2^{12}. A_3^{8}, A_4^{6}, A_6^{4}, A_8^{3}, A_{12}^{2}, A_{24},

D_4^{6}, D_6^{4}, D_8^{3}, D_{12}^{2}, D_{24}, E_6^{4}, E_8^{3}

$A_5^{4}.D_4$, $A_7^{2}\,D_5^{2}$, $A_9^{2}.D_6$, $A_{15}D_9$,

$E_6\,D_7\,A_{11}$, $E_7^{2}\,D_{10}$, $E_7\,A_{17}$, $E_8\,D_{16}$

Thm In each 24 case there is a unique Λ with the root system listed above.!

proof: In case $\Lambda(2) = \phi$, we have the famous Leech lattice.

Let M be one of the 23 nonzero root lattices listed above

Suppose Λ even self dual s.t. $M(2) = \Lambda(2)$. Then

$M \subseteq \Lambda \subseteq M^\vee$ and $|\Lambda/M| = |M^\vee/\Lambda| = k$ for some

small k. So Λ/M is a finite abelian grp of order k in the finite abelian gp $(M^\vee/M)$. One constructs this finite group by hand and shows that there is an unique choice in each case. These are called Niemeier lattices $\square$

Puzzle: Explain why the above case by case construction works, ie find an uniform existence proof of the 23 Niemeier lattices

Ref. [Conway and Sloane] Sphere packings, lattices and groups.

[Milnor and Husemoller] Symmetric bilinear forms.

[Serre] Course in arithmetic