G-quadratic, LG-quadratic, Koszul Quotients of Exterior Algebras

(joint w/ Zach Mere)
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$R = \bigoplus_i R_i$ a $\mathbb{Z}$-graded K-algebra
(K a field)

M = f.g. graded R-module
$HF(M, i) = \dim_K (M_i)$
$HS_M(t) = \sum_i HF(M,i)\, t^i$

e.g. $E = \Lambda_K \langle e_1, \ldots, e_n \rangle$
(exterior algebra)

$HS_E(t) = (1+t)^n$

$HF(E, i) = \binom{n}{i}$

M has a graded free resolution $F_\bullet$ over R, where

$$F_i = \bigoplus_j R(-j)^{\beta_{ij}(M)}$$

← graded Betti #'s

where $R(-j)_i = R_{i-j}$

$$\beta_i(M) = \sum_j \beta_{ij}(M)$$

total Betti numbers

Poincare Series

$$P_M^R(t) = \sum_{i \geq 0} \beta_i(M) t^i$$

If R is commutative, $x \in R_1$ is <u>regular</u> on M if it is a nzd on M. $\{m \in M \mid xm = 0\} = \{0\}$

If $R = E$, $x \in R_1$ is <u>regular</u> on M if $\{m \in M \mid xm = 0\} = xM$

(Recall $x^2 = 0$. Best we can hope for)

Equivalently $M \xrightarrow{x} M \xrightarrow{x} M$ is exact.

If x is not regular, it is M-singular.

The set of M-singular elements in E_1, denoted $V_E(M)$, is called the singular variety of M.

Facts: 1. $V_E(M)$ is a union of linear subspaces of E_1.

2. $M \leq N \implies$ each $V_E(M), V_E(N), V_E(N/M)$

is contained in union of other 2.

A sequence $x_1, \ldots, x_n \in \mathcal{E}$ is reg on M if x_i is regular on $M/(x_1, \ldots, x_{i-1})M$. $\forall i$

$\operatorname{depth}_{\mathcal{E}}(M)$ = length of maximal regular sequence.

Fix $\leq$.

Given an ideal $I \subseteq R$, $g_1, \ldots, g_t \in I$ is a <u>Gröbner basis</u> if $\langle LT(g_1), \ldots, LT(g_t) \rangle = \langle LT(f) \mid f \in I \rangle$

Fact: $HS_{\mathcal{E}/I}(t) = HS_{\mathcal{E}/\mathrm{In}(I)}(t)$

A positively graded K-algebra R is <u>Koszul</u> if K has a linear free R-resolution, i.e.

$$\beta_{ij}^R(K) = 0 \quad \text{for } i \neq j$$

<u>Facts</u>: ① $S = K[x_1, \dots, x_n]$

$E = \Lambda_K\langle e_1, \dots, e_n\rangle$

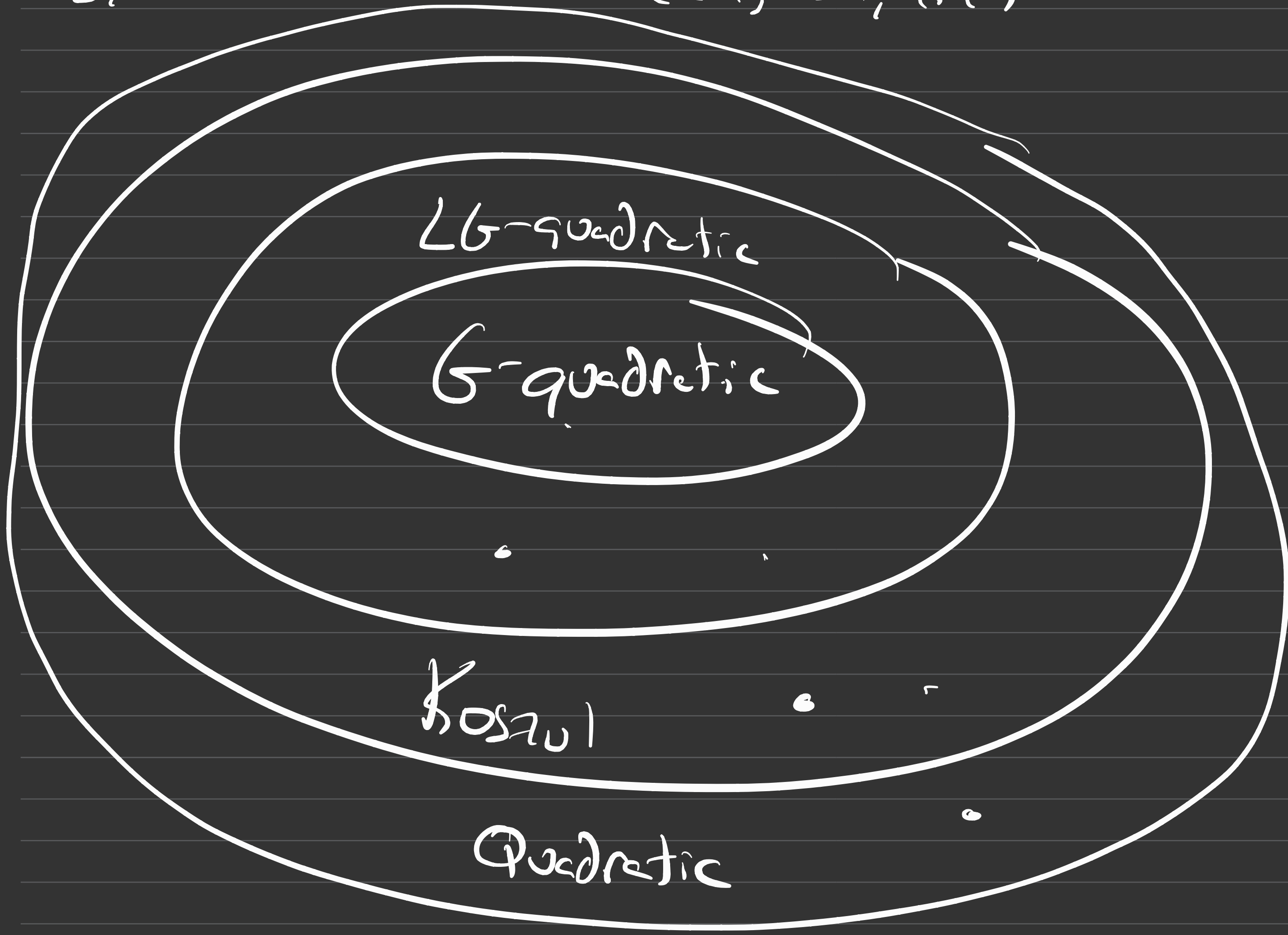
are Koszul.

I has a GB of quads $\overset{\Longrightarrow}{\not\Longleftarrow}$ $\frac{S}{I}, \frac{E}{I}$ Koszul $\overset{\Longrightarrow}{\not\Longleftarrow}$ I gen by quads.

"G-quadratic"

R is LG-quadratic if $\exists$ a G-quadratic algebra A and a regular sequence $x_1, \dots, x_n$ on A

s.t. $R \cong A/(x_1, \dots, x_n)$



$\boxed{Q1}$ Is there an LG-quadratic quotient of an exterior algebra that is not G-quadratic?

$\boxed{A1}$ Yes.

$$R = \frac{\Lambda_K\langle e_1, \dots, e_7\rangle}{(e_1e_2, e_2e_3, e_3e_4, e_4e_5, e_5e_6, e_6e_7)}$$

Monomial + quadratic, G-quadratic

Check $x = e_1 + e_4 + e_7$ is regular on R. So $R/(x)$ is LG-quadratic.

$$HS_{R/(x)}(t) = 1 + 6t + 9t^2 + t^3$$

No ideal gen by dg. 2 monomials with this HS. So $R/(x)$ can't be G-quadratic.

$\boxed{\text{Q2}}$ Is there a Koszul quotient of an exterior algebra that is not LG-quadratic?

$\boxed{A2}$ Yes

<u>Thm</u> (Fröberg-Löfwall)
$\mathrm{Char}(k)=0$, $E=\Lambda_k\langle e_1,\dots,e_n\rangle$
$I=(f_1,\dots,f_t)$ gen by t generic quadrics. If $t \geq \binom{n}{2}-\frac{n^2}{4}$, then E/I is Koszul.

<u>Thm (MM)</u> 6 generic quadrics in 6 exterior variables define a Koszul but not LG-quadratic quotient.

Idea: Koszulness ✓

Identify a quadric $q \in E_2$ with an alternating matrix A

so
$$q = \underline{e} A \underline{e}^T$$

e.g. $\alpha e_1 e_2 + \beta e_1 e_3 + \gamma e_2 e_3$

$$\longleftrightarrow \quad \frac{1}{2} \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}$$

$\mathrm{rank}(q) := \mathrm{rank}(A) = \text{even}.$

__Prop:__ $\mathrm{rank}(q) \geq 2r \quad \forall\, 0 \neq q \in I$

iff $\quad t \leq \dfrac{(n-2r+1)(n-2r+2)}{2}$

__Proof:__ $\mathrm{rank}(q) < 2r$

$\iff q \in V(J)$

$$J = I_{2r}(A) \subseteq \mathbb{P}_K^{\binom{n}{2}-1}$$

$\uparrow$ generic alternating $n \times n$ matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & \cdots \\ -x_1 & 0 & & \\ -x_2 & & 0 & \\ \vdots & & & 0 \\ & & & & 0 \end{pmatrix}$$

$\sqrt{J} = Pf_{2r}(A)$

ideal of $2r \times 2r$ Pfaffians of A

$(\det(A) = f^2, \ f = Pf(A))$

(Aside $S/Pf_{2r}(A)$

$= K[Y^T \Omega Y]$

$= K[Y]^{Sp_{2r-2}(K)}$

$Y =$ generic $(2r-2) \times n$ matrix

$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$)

$\operatorname{codim}(J) = \operatorname{codim}(Pf_{2r}(A))$

$$= \frac{(n-2r+1)(n-2r+2)}{2}$$

t generic quadrics

$\rightsquigarrow$ t generic points in $\mathbb{P}^{\binom{n}{2}-1}$

which span a linear space L

of $\dim = t-1$

If $t \leq \frac{(n-2r+1)(n-2r+2)}{2}$

then $L \cap V(J) = \emptyset$. $\square$

<u>Cor 1</u>: Every nonzero quadric in $I = (q_1, \ldots, q_6) \subseteq E$ has rank at least 4. ($r=2$ above)
generic

<u>Cor 2</u>: If $q_1, q_2 \in \Lambda_K\langle e_1, \dots, e_4\rangle$ are lin. ind quadrics, $\exists\, q \in (q_1, q_2)$ with $\operatorname{rank}(q) = 2$.

$$HS_{E/I}(t) = 1 + 6t + 9t^2$$
$$= (1+3t)^2$$

If no edge ideal with same HS, done.

However,

$$HS_{E/I}(\triangle\triangle) = (1+3t)^2$$

Done by proving:

<u>Lemma</u>: If $\mathrm{In}_<(I) = I(\triangle\triangle)$ then I contains a rank 2

quadric.

Proof: Technical.

Illuminating Example

$$I = (e_1e_2 + e_3e_4,\ e_1e_3 + e_2e_4,$$
$$e_2e_3 + e_1e_4,\ e_5e_6 + e_7e_8,$$
$$e_5e_7 + e_6e_8,\ e_6e_7 + e_5e_8)$$

$$HS = (1+3t)^2(1+t)^2$$

It is a revlex GB

But $(e_1e_2+e_3e_4)+(e_1e_3+e_2e_4)$

$$= (e_1+e_4)(e_2+e_3).$$

rank 2.