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From Lie algebras to Lie superalgebras Part II

Recap: Lie algebras — algebraic structure, geometry, analysis } plenty of details left out

e.g. explicit manifold language

super vector spaces — "super spaces", $\mathbb{Z}_2$-grading & parity, super linear algebra

$$V = V_{\bar{0}} \oplus V_{\bar{1}} = \mathrm{span}\begin{pmatrix}\text{even}\\ \text{basis}\end{pmatrix} \oplus \mathrm{span}\begin{pmatrix}\text{odd}\\ \text{basis}\end{pmatrix}$$

$\{$linear maps on $V\}$ — super space itself, parity preserving/reversing

Goal: introduce $\mathfrak{osp}(1|2)$

&

some of its representations

From LA to LSA

Start with $\mathbb{C}$

Think of a symmetric form

$$\beta_s(x,y) = \beta_s(y,x)$$

If β_s nondegenerate & bilinear, then $\beta_s(x,y)$ "=" xy

Start with $\mathbb{C}^2$

How about a nice ~~skew~~-symmetric form on $\mathbb{C}^2$

anti-

$$\beta_A(x,y) = -\beta_A(y,x)$$

If β nondegenerate & bilinear, then $\beta(x,y)$ "=" $x_1 y_2 - y_1 x_2$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\mathbb{C}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Start with $\mathbb{C}^{1|2}$

and a form $\beta : \mathbb{C}^{1|2} \times \mathbb{C}^{1|2} \to \mathbb{C}$

$\boxed{\text{even}}$ $\beta(x,y) = 0$ if $|x| \neq |y|$

$\boxed{\text{supersymmetric}}$ $\beta(x,y) = (-1)^{|x||y|}\beta(y,x)$

$\boxed{\text{nondegenerate}}$ & $\boxed{\text{bilinear}}$

There's a standard choice for β

More appropriately, we can encode β as

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

Which maps/matrices preserve $(\mathbb{C}, \beta_s)$?

Group: $\beta_s(Ax, Ay) = \beta_s(x,y)$

$t \mapsto e^{tm}$, m: 1-by-1 matrix

$$(e^{tm})^T [\beta_s] e^{tm} = [\beta_s]$$

$[1]$

$$\rightarrow ((e^{tm})^T)' [\beta_s] e^{tm} + (e^{tm})^T([\beta_s](e^{tm})') = 0$$

$$m^T[\beta_s] + [\beta_s]m = 0$$

$$m = 0$$

Repeat with β_A & M: 2-by-2 matrix

$$M^T[\beta_A] + [\beta_A]M = 0$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

r, s, a, b, c complex #s

So which matrices preserve $(\mathbb{C}^{1|2}, \beta)$

$$\left[\begin{array}{c|cc} 0 & r & s \\ \hline s & a & b \\ -r & c & -a \end{array}\right]$$

LSA with

$$[x,y] = xy - (-1)^{|x||y|} yx$$

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$\mathfrak{osp}(1|2) =$ odd space $\oplus$ even space

$$\mathbb{C}\left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \oplus \mathbb{C}\left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \oplus$$

$$\mathbb{C}\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right] \oplus \mathbb{C}\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \oplus \mathbb{C}\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Representations of $osp(1|2)$

- Let's back-up

"linearization of group action"

$$osp(V, B) = \left\{ \begin{array}{l} \text{set of maps on } V \\ \text{preserving orthosymplectic space} \end{array} \right\}$$

$OSp(V, B)$: the supergroup

$$osp(\mathbb{C}^{1|2}, B) \rightsquigarrow osp(1|2)$$

standard representation 3×1 column vectors

let

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \dot{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \dot{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \dot{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Computation</u>

$$[X, Y] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (-1)^{|1||1|} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \; \end{bmatrix}$$

Important algebras in representation theory & $osp(1|2)$

$T(\mathbb{C}^{1|2})$

is defined

the same

Definition 3.2.3. The k^{th} *tensor power of* V is

$$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{k factors}},$$

with $T^0(V) = \mathbb{C}$ and $T^1(V) = V$. Elements of the vector space $T^k(V)$ are called *k-tensors.*

Definition 3.2.4. The *tensor algebra* $T(V)$ *on* V is the associative algebra

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V),$$

with associative product $T^m(V) \otimes T^n(V) \to T^{m+n}(V)$ defined on m-tensors and n-tensors by $(v_{i_1} \otimes \cdots \otimes v_{i_m}) \otimes (v_{j_1} \otimes \cdots \otimes v_{j_n}) = v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v_{j_1} \otimes \cdots \otimes v_{j_n}$, and extended by linearity.

Another formulation of $T(V)$ is as the algebra of complex polynomials with basis elements of V serving as noncommuting indeterminates.

Proposition 3.2.5. *The tensor algebra on $T(V)$ is identified with the free algebra $\mathbb{C}\langle B \rangle$, where B is any basis of V.*

Remark 3.2.6. We will write products as either $a \otimes b$ or ab depending on emphasis.

More algebras

$$\underbrace{(V,\beta_A)}$$

Form $T(V)$ for a symplectic space

and quotient out by ideal generated

by $J = \{ x\otimes y - y\otimes x - \beta_A(x,y) \}$

$T(\mathbb{C}^2)/J$ is the first Weyl algebra,

which can be presented as

the algebra $\mathcal{D}(1)$ of polynomial differential operators on $\mathbb{C}[x]$

Also $\langle D, M\rangle_{Alg}$

$D : \mathbb{C}[x] \to \mathbb{C}[x]$

$f \mapsto \frac{df}{dx}$

$DM - MD = 1_{\mathbb{C}[x]}$

$M : \mathbb{C}[x] \to \mathbb{C}[x]$

$g \mapsto xg$

Universal enveloping algebra of Lie algebra $\mathfrak{g}$

Form $T(\mathfrak{g})$ and quotient out the ideal generated by

$$I = \{ x \otimes y - y \otimes x - [x,y] \}$$

similarly, $T(osp(1|2))/I$ = the universal enveloping algebra $U(osp(1|2))$ of $osp(1|2)$

Tensor product representation

Known (see Musson e.g.) surjective map of associative algebras

$$\begin{tikzcd} \mathfrak{osp}(1|2) \arrow[r, hook] \arrow[rrr, bend right] & \mathcal{U}(\mathfrak{osp}(1|2)) \arrow[r, two heads] & \mathcal{D}(1) \arrow[r, hook] & \mathrm{End}(\mathbb{C}[x]) \end{tikzcd}$$

Conclude $\mathfrak{osp}(1|2)$ acts on $V = \mathbb{C}[x] \otimes \mathbb{C}^{1|2}$

$$\frac{1}{\sqrt{2}}\frac{d}{dx} \longleftarrow\!\shortmid X \longmapsto \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$$\frac{1}{\sqrt{2}}x \longleftarrow\!\shortmid Y \longmapsto \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x\frac{d}{dx} + \frac{1}{2} \longleftarrow\!\shortmid H \longmapsto \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

$$\frac{1}{2}x^2 \longleftarrow\!\shortmid E \longmapsto \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^2 \longleftarrow\!\shortmid F \longmapsto \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

LSA actions on tensor products

$$g\cdot(v\otimes w) = g\cdot v\otimes w + \underbrace{(-1)^{|g||v|}}_{\text{super}} v\otimes g\cdot w$$

$$y\cdot\left(x^2\otimes\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = ?$$

$$\frac{1}{\sqrt{2}}x^3\otimes\begin{bmatrix}1\\0\\0\end{bmatrix} + x^2\otimes\begin{bmatrix}0\\1\\0\end{bmatrix}$$

New Let's call $V = \mathbb{C}[x_1,\dots,x_n]\otimes\mathbb{C}^{1|2n}$

- The superspace V decomposes as a direct sum of simple $\mathfrak{osp}(1|2n)$-representations with explicit formulas

$\mathfrak{osp}(1|2n)$ acts on poly $\otimes$ ext. pwr

- $\mathbb{C}[x_1,\dots x_n]\otimes\underline{\mathbb{C}[\xi_1,\dots,\xi_{2n}]}$, ξ_i anti-commute

view as $\underline{\Lambda^0(\mathbb{C}^{2n})\oplus\Lambda^1(\mathbb{C}^{2n})}$