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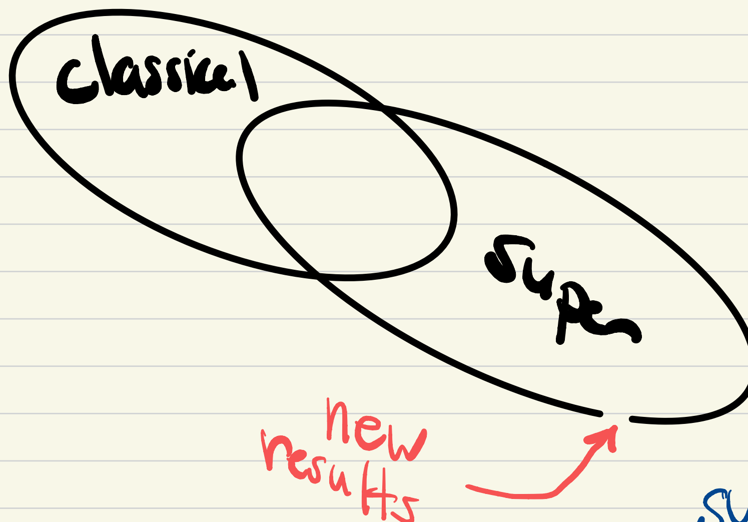
☆ Alg & Geo Seminar ☆

From Lie algebras to Lie superalgebras

definitions

examples



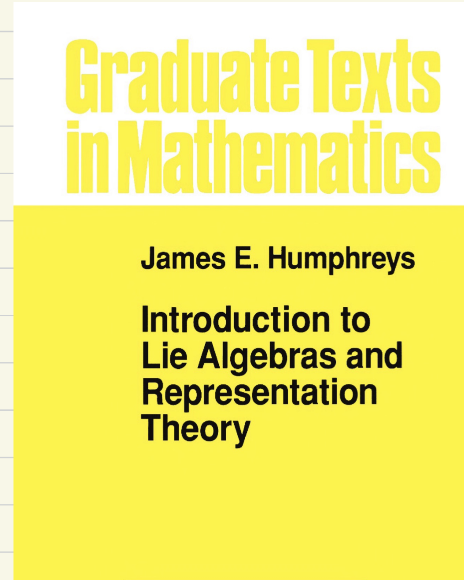
Goals

- explore various entry points to studying Lie super algebras
- set up talk #2
- math in community

Where to begin?

algebraically

Choose your favorite vector space



$V = \mathbb{k}[x]$ as a $\mathbb{k}$-vector space

$\mathbb{k}$: field

$\mathcal{L}(V) = \{\text{linear transformations } T: V \to V\}$

$D: \mathbb{R}[x] \to \mathbb{R}[x]$

$f \mapsto \frac{df}{dx}$

is an element of $\mathcal{L}(\mathbb{R}[x])$

$\mathcal{L}(V)$ is a $\mathbb{k}$-vector space

<u>AND</u> function composition on $\mathcal{L}(V)$

is a generally non-commutative product

$\underbrace{(DM - MD)}_{\text{Identity}_{\mathbb{R}[x]}}(f) = f$

$D(xf) - x\frac{df}{dx} =$

$f + x\frac{df}{dx} - x\frac{df}{dx} =$

So X & Y in $\mathcal{L}(V) \overset{\text{often}}{\Longrightarrow} XY - YX$ in $\mathcal{L}(V) \setminus \{0\}$

$[\partial_{x_i}, x_j] = \delta_{ij} \mathbb{1}_{\mathbb{R}[x]}$

Now let $V = \mathbb{R}^3$.

Then we can "encode" $\mathcal{L}(V)$

as $\{3\times 3 \text{ matrices with entries in } \mathbb{R}\}$

$\mathcal{L}(V) \cong \mathrm{Mat}(3,\mathbb{R})$ as vector spaces

choice of basis

Still, $\mathrm{Mat}(3,\mathbb{R})$ has a generally non-commutative product

$A \,\&\, B$ in $\mathrm{Mat}(3,\mathbb{R}) \Rightarrow \underbrace{AB-BA}_{\text{Commutator}}$ in $\mathrm{Mat}(3,\mathbb{R})$

BUT

$\mathbb{R}^3$ has its own product — cross product

with properties: bilinearity, alternativity, non-associativity & Jacobi identity

Definition of Lie algebra

Fraktur $\mathfrak{g}$: vector space over field $\mathbb{F}$

$[\alpha x + y, \beta v + w] = \alpha\beta[x,v] + \alpha[x,w] + \beta[y,v] + [y,w]$

v, w, x, y vectors & α, β scalars

Definition 3.3.1. A Lie algebra is a pair $(\mathfrak{g}, [\ ,\ ])$ such that $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear product (called the *Lie bracket*) on $\mathfrak{g}$ and the following properties hold for all vectors $x, y, z \in \mathfrak{g}$.

char$(\mathbb{F}) \neq 2$

(3.4) $$[x, y] = -[y, x]. \iff [x,x] = 0, \text{ alternativity}$$

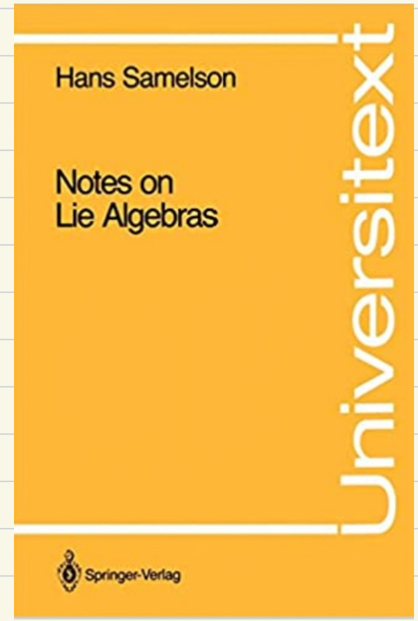
(3.5) $$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \text{ Jacobi identity}$$

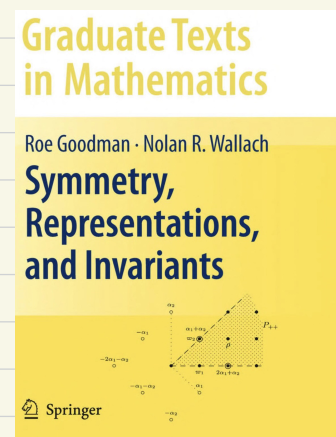
Another look

more geometrically (invariantly)

forms on vector space
give geometry

certain maps take points/vectors
in general linear position to
points to general linear position





So we can ask ... given some form

$\beta : V \times V \to \mathbb{F}$

which invertible transformations preserve the form β

$\{ A \in GL(V) \mid \beta(Ax, Ay) = \beta(x, y), \text{ for all vectors } x \,\&\, y \}$

Wait...

now we're talking matrix groups for $V \cong \mathbb{F}^n$, say $\mathbb{C}^n$

Name	**Notation**		**Defining Condition**	
Special linear group	$\mathrm{SL}(\mathbb{C}^n)$	$\mathrm{SL}(n)$	Preserves oriented volume	$\det(A) = 1$
Symplectic group	$\mathrm{Sp}(\mathbb{C}^{2n})$	$\mathrm{Sp}(2n)$	Preserves symplectic form	$A^T J_{2n}^{skew} A = J_{2n}^{skew}$
Orthogonal group	$\mathrm{O}(\mathbb{C}^n)$	$\mathrm{O}(n)$	Preserves nonsingular quadratic form	$A^T J_n^{sym} A = J_n^{sym}$
Special orthogonal group	$\mathrm{SO}(\mathbb{C}^n)$	$\mathrm{SO}(n)$	Preserves nonsingular quadratic form and oriented volume	$A^T J_n^{sym} A = J_n^{sym}$ $\det(A) = 1$

We can encode our forms into matrices

$$J_{2n}^{skew} = \overset{2n \times 2n}{\left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]}, \quad \dim(V) = 2n, \quad I_n \text{ the } n \times n \text{ identity matrix}$$

$$J_{2n}^{sym} = \left[\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right], \dim(V) = 2n, \text{ and}$$

$$J_{2n+1}^{sym} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_n \\ \hline 0 & I_n & 0 \end{array}\right], \dim(V) = 2n+1$$

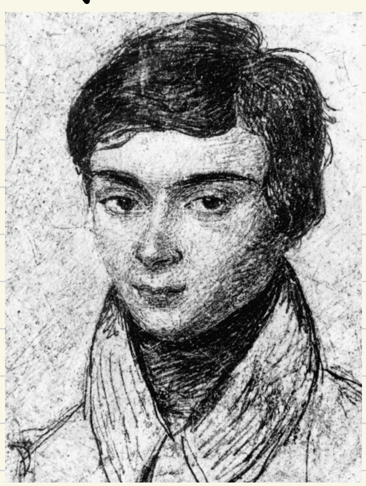
In particular we have examples of Lie groups

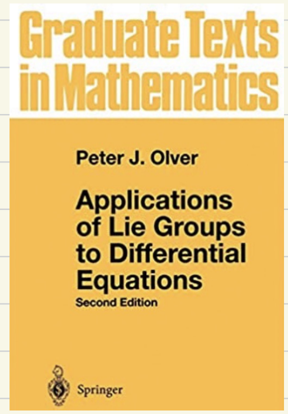
roots of polynomials

&

finite groups

Galois





solutions of differential equations

&

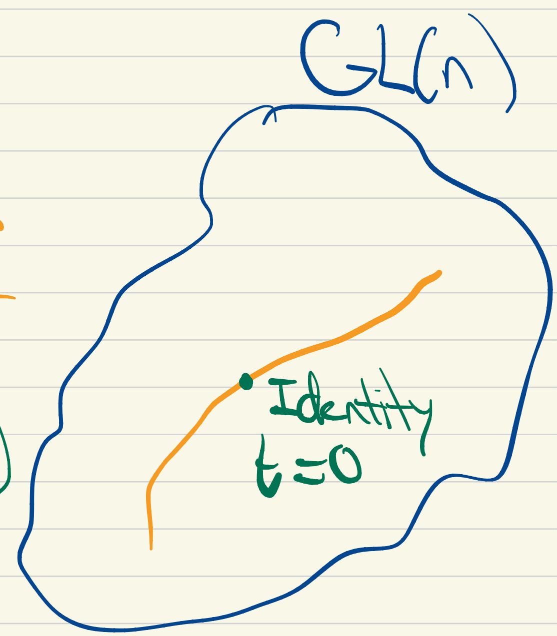
continuous transformation groups

Working over $\mathbb{C}$

Exponentiation shows up in solving ordinary differential equations

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix} & \cdot & \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

exp: $\underline{Mat(n)} \twoheadrightarrow GL(n)$



exponential map $\quad A \longmapsto e^{tA} = \sum_{k\in\mathbb{N}_0} \frac{(tA)^k}{k!}$

for a parameter t

($n\times n$ identity when $k=0$)

So which matrices map to $Sp(n)$, $O(n)$, $SO(n)$

Some Lie algebras from Lie groups

Definition 3.3.21. The *symplectic Lie algebra* $\mathfrak{sp}(2n) \subset \mathfrak{gl}(2n)$ is the set of $2n \times 2n$ matrices M such that $\underline{J_{2n}^{skew} M + M^T J_{2n}^{skew}} = 0$. We can express each element M of $\mathfrak{sp}(2n)$ as a block matrix with $n \times n$ matrices A, B, C:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right]$$

$\beta(Av, Aw) = \beta(v, w)$

$sp(2) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

B, C: symmetric matrices

$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Definition 3.3.22. The *orthogonal Lie algebra* $\mathfrak{so}(2n) \subset \mathfrak{gl}(2n)$ is the set of $2n \times 2n$ matrices M such that $J_{2n}^{sym} M + M^T J_{2n}^{sym} = 0$. We can express each element of $\mathfrak{so}(2n)$ as a block matrix with $n \times n$ matrices A, B, C:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right]$$

where B and C are skew-symmetric matrices.

$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Also, $sl(2)$ where we $tr(\text{matrix}) = 0$

In super case, orthogonal & symplectic can be considered simultaneously

Definition 3.3.23. The *orthogonal Lie algebra* $\mathfrak{so}(2n+1) \subset \mathfrak{gl}(n+1)$ is the set of $(2n+1) \times (2n+1)$ matrices M such that $J_{2n+1}^{sym} M + M^T J_{2n+1}^{sym} = 0$. For completeness, $\mathfrak{so}(1) = 0$. We can express each element M of $\mathfrak{so}(2n+1)$ as a block matrix with $n \times n$ matrices A, B, C and $1 \times n$ row vectors r, s:

$$M = \left[\begin{array}{c|c|c} 0 & r & s \\ \hline -s^t & A & B \\ \hline -r^T & C & D \end{array}\right],$$

where B and C are skew-symmetric matrices

SUPER SPACE

- careful: I'm not a geometer, not a physicist

super space means super vector space

$$V = V_{even} \oplus V_{odd} = V_{\bar{0}} \oplus V_{\bar{1}}$$

$\mathbb{Z}_2$-graded vector space

ex) $\mathbb{C}^3 = \mathbb{C}^1 \oplus \mathbb{C}^2$
$\quad = \mathbb{C}^0 \oplus \mathbb{C}^3$

nothing new in world of vector spaces

In the category of super vector spaces,

$$\mathbb{C}^{1|2} \neq \mathbb{C}^{0|3}$$

Parity (V: $V = V_{\bar 0} \oplus V_{\bar 1}$ super vector space)

parity map $p: \underbrace{V_{\bar 0} \sqcup V_{\bar 1}}_{\text{homogeneous elements of } V} \to \mathbb{Z}_2$

$v \mapsto \bar 0$, if $v \in V_{\bar 0} \setminus \{0\}$

$v \mapsto \bar 1$, if $v \in V_{\bar 1} \setminus \{0\}$

$p(x) = |x|$

Definition of Lie superalgebra

Begin with a super vector space V

Consider

$$\underbrace{\mathfrak{gl}(V)}_{\text{general linear Lie superalg}} \leftarrow \underbrace{\{T: V \to V\}}_{\text{set of all linear maps}} = \{\text{parity preserving}\} \oplus \{\text{parity reversing}\}$$

Any linear map T on V decomposes as the unique sum of a parity-preserving map with a parity-reversing map

super case (cat. of super-vector spaces)

morphisms preserve parity

$\cancel{\text{normally } \mathbb{C}^3 \to \mathbb{C}^3, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}}$

$T: \mathbb{C}^{1|2} \to \mathbb{C}^{1|2}$

$T(\text{even}) \subseteq \text{even}$