Uspekhi Mat. Nauk **40**:1 (1985), 29–66 Russian Math. Surveys **40**:1 (1985), 31–75

Hyperbolic reflection groups

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Contents

Introduction

1. Let X^n be the n-dimensional Euclidean space E^n, the n-dimensional sphere S^n, or the n-dimensional Lobachevskii space Λ^n.

A convex polytope $P \subset X^n$ bounded by hyperplanes H_i, $i \in I$, is said to be a *Coxeter polytope* if for all $i, j \in I$, $i \neq j$, the hyperplanes H_i and H_j are disjoint or form a dihedral angle of π/n_{ij}, where $n_{ij} \in \mathbf{Z}$ and $n_{ij} \geqslant 2$. (We have in mind a dihedral angle containing P.)

If P is a Coxeter polytope, then the group Γ of motions of X^n generated by the reflections R_i in the hyperplanes H_i is discrete, and P is a fundamental polytope for it. This means that the polytopes γP, $\gamma \in \Gamma$, do not have pairwise common interior points and cover X^n; that is, they form a tessellation for X^n. The relations

$$R_i^2 = 1, \quad (R_i R_j)^{n_{ij}} = 1 \tag{1}$$

are defining relations for Γ. (If H_i and H_j are disjoint, we set $n_{ij} = \infty$; in this case there is no relation between R_i and R_j.)

Conversely, every discrete group Γ of motions of X^n generated by reflections in hyperplaces can be obtained in the manner just described. Here P can be taken as any of the convex polytopes into which X^n is partitioned by the mirrors of all the reflections belonging to Γ.

In what follows, "reflection" always means "reflection in a hyperplane" and "reflection group" means "group generated by reflections". Discrete reflection groups in Euclidean space (and thus on the sphere) have been studied by Weyl [9], Cartan [26], [27], and Coxeter [28], [29]. In particular, a complete classification was obtained. These groups are well known and they plan an important part in the theory of semisimple Lie groups [6], [31].

An abstract group Γ with generators R_i and defining relations (1) is said to be a *Coxeter group*. As Tits has proved ([6], [58], [59]), every finitely generated Coxeter group can be represented as a group of projective maps generated by reflections and acting discretely in some domain of projective space. An algebraic description of all representations of this form has been obtained by the author [12], [13].

However, among the "non-Euclidean" Coxeter groups the *hyperbolic* ones are of greatest interest, that is, those that can be represented as discrete reflection groups in a Lobachevskii space; and among the hyperbolic groups above all those whose fundamental polytopes have finite volume. This paper is devoted to a study of these groups.

2. We call discrete reflection groups with fundamental polytopes of finite volume *crystallographic reflection groups* (c.r.g. for short).

The c.r.g. in the Lobachevskii plane were described as long ago as 1882 by Poincaré in a memoir on Fuchsian groups [52], and by von Dyck [23]. The fundamental polytope of such a group may have an arbitrary number of sides and angles $\pi/n_1, \ldots, \pi/n_k$ (where n_i is either ∞ or an integer $\geqslant 2$), provided only that

$$\frac{1}{n_1}+\ldots+\frac{1}{n_k}<k-2$$

Thus, it depends on $k-3$ independent parameters.

The Lobachevskii plane occupies an exceptional position. As follows from the theorem on the strong rigidity of discrete subgroups of $O_{n,1}$ ([45], [51]), or from results of Andreev [2] (for bounded polytopes), a Coxeter polytope of finite volume in a Lobachevskii space of dimension at least 3 is determined by its dihedral angles. Therefore, there are at most countably many c.r.g. in Lobachevskii spaces of dimension at least 3.

Up to 1965, only comparatively few examples of such groups were known. These examples arose in the four following directions of research.

1°. The tessellation of space by equal regular polytopes.
The group of symmetries of such a tessellation is a c.r.g. As a fundamental polytope one can take a simplex with vertices at the centres of an increasing sequence of faces of all dimensions of some tessellating polytope. All the tessellations of Lobachevskii spaces by bounded equal regular polytopes were found by Schlegel [63] in 1883, those by unbounded polytopes (but still of finite volume) by Coxeter [30] in 1954 (see also [32]).

2°. Homogeneous simplicial complexes.
In 1950, Lannér [33] investigated the simplicial complexes whose underlying topological spaces are simply-connected n-dimensional manifolds and which allow automorphism groups containing (combinatorial) reflections in all $(n-1)$-dimensional simplexes and acting simply transitively on the set of n-dimensional simplexes. He listed all such complexes and obtained geometrical realizations of them in spaces of constant curvature. In particular, he found all c.r.g. in Lobachevskii spaces whose fundamental polytopes are bounded simplexes. There are seven such groups in Λ^3 and five in Λ^4 (see Table 3 on p.60). Six of them are connected in the way described above with tessellations into regular polytopes.

It is not difficult to list all c.r.g. in Lobachevskii spaces whose fundamental polytopes are unbounded simplexes. There are some ten groups of this sort in the spaces Λ^n, $n \geqslant 3$, the maximum value of n being 9 ([6], [10]).

3°. Clifford-Klein space forms.
The problem of classifying space forms was posed by Klein in 1880. These forms are none other than Riemannian manifolds of the form X^n/Γ, where $X^n = E^n$, S^n, or Λ^n, and Γ is a discrete group of motions with fixed-point-free action.. For a long time it was not known whether there exists compact space forms of negative curvature of dimension at least 3. Apparently the first examples of such manifolds were given in 1931 by Löbell [34]. His starting point was the construction in three-dimensional Lobachevskii space of a 14-gon P (similar to a dodecahedron but with hexagonal bases) with dihedral right angles. The three-dimensional manifolds found by Löbell are spaces of the form Λ^3/Γ, where Γ is a subgroup of finite index in the group generated by reflections in the planes bounding P.

Two years later Weber and Seifert [8] constructed "the hyperbolic space of the dodecahedron", which has since become more widely known. It can also be obtained by forming the factor group of Λ^3 by a subgroup of finite index in a c.r.g., namely, a subgroup of index 120 in the symmetry group of the tessellation of Λ^3 into regular dodecahedra.

4°. Arithmetic groups.
In [61] Fricke, regarding the automorphism groups of indefinite ternary integral quadratic forms as discrete groups of motions of the Lobachevskii plane, showed that in certain cases they contain a subgroup of finite index generated by reflections, and found fundamental polytopes of these groups.

In 1892 Bianchi [7] applied Fricke's method to the groups $PGL_2(A_m) \rtimes \langle\tau\rangle$, where A_m is the ring of integers in the imaginary quadratic field $K_m = \mathbf{Q}(\sqrt{-m})$, the symbol $\rtimes$ denotes the semi direct product, and τ is complex conjugation. Regarding these groups as discrete groups of motions of three-dimensional Lobachevskii space, he proved that for $m \leqslant 19$, $m \neq 14, 17$ they contain a subgroup of finite index (1 or 2) generated by reflections. He gave an explicit description of the fundamental polytopes of the c.r.g. obtained in this way.

Examples of arithmetically defined c.r.g. in Λ^3 are also in the monograph [62] of Fricke and Klein. However, they are all commensurable with Bianchi groups.

3. In 1965–1966 Makarov ([36]–[39]) proposed certain geometric constructions that open up unlimited possibilities for constructing new examples of c.r.g. in Lobachevskii space Λ^3, among them some that contain elements of any given order, plane c.r.g., *etc*. These papers by Makarov and the interest in a geometric construction of discrete groups of motions, which stems from the problem whether discrete subgroups of semisimple Lie groups are arithmetic, stimulated the development of a theory of c.r.g. in Lobachevskii spaces.

From a geometric point of view the most important property of Coxeter groups is that they are *acute-angled* (for the definition see §2.1). In 1970 Andreev ([2], [3]) gave an exhaustive description of acute-angled polytopes of finite volume in three-dimensional Lobachevskii space. He indicated simple necessary and sufficient conditions under which there is in Λ^3 a convex polytope of finite volume with given dihedral angles not exceeding $\pi/2$ (for example, fractions of π). In effect he obtained thereby a *classification of the c.r.g. in* Λ^3.

Since acute-angled spherical polytopes not containing diametrically opposite points are simplexes ([27], [28]), for bounded acute-angled polytopes in Euclidean or Lobachevskii space the polyhedral angles at the vertices are simplicial. Convex polytopes of this combinatorial structure are said to be simple. The combinatorial type of a simple convex polytope is said to be simple.

The following are the Andreev conditions for the existence in Λ^3 of a bounded acute-angled polytope of given simple combinatorial type other than a simplex or a triangular prism, with given dihedral angles:

1) if three faces meet at a vertex, then the sum of the dihedral angles between them is greater than π;

2) if three faces are pairwise adjacent, but not concurrent, then the sum of the dihedral angles between them is less than π;

3) if four faces are "cyclically" adjacent (like the side faces of a quadrilateral prism), then the sum of the dihedral angles between them is less than 2π (that is, not all of them are $\pi/2$).

All of these conditions are in the form of (linear) inequalities. This is explained by the fact that in Λ^3 the number of degrees of freedom for a simple convex polytope of given combinatorial structure is equal to the number of its dihedral angles (that is, the number of edges). For simple convex polytopes in the spaces Λ^n with $n \geqslant 4$ the number of dihedral angles can be larger than the number of degrees of freedom[(1)]. Nevertheless, in these cases there may exist Coxeter polytopes of given combinatorial type. (In fact, the examples in §5.5 and §5.6 of this paper are of this type.) Therefore, one cannot hope for a classification of c.r.g. in the Λ^n for $n \geqslant 4$ of the same kind as there is for Λ^3.

In unbounded acute-angled polytopes of finite volume in Λ^3 four faces may meet in vertices at infinity. The Andreev conditions for such polytopes (other than a simplex of a triangular prism) can be stated as follows:

1) if three faces meet at a vertex, then the sum of the dihedral angles between them is at least π;

2) if four faces meet at a vertex, then the sum of the dihedral angles between them is 2π (that is, they are all $\pi/2$);

3) if three faces are pairwise adjacent but not concurrent at a vertex, then the sum of the dihedral angles between them is less than π;

4) if a face Γ_1 is adjacent to faces Γ_2 and Γ_3, while Γ_2 and Γ_3 are not adjacent, but have a common vertex not in Γ_1, then the sum of the dihedral angles formed by Γ_1 with Γ_2 and with Γ_3 is less than π;

5) if four faces are cyclically adjacent, but do not meet at a vertex, then the sum of the dihedral angles between them is less than 2π (that is, they are not all $\pi/2$).

4. The c.r.g. in Lobachevskii spaces and the groups commensurable with them are a very special form of discrete groups of motions with fundamental domains of finite volume. For example, in a Lobachevskii space of odd dimension the discrete arithmetic groups connected with non-commutative algebraic extensions of $\mathbf{Q}$ are never commensurable with c.r.g. [10]. Moreover, for any n there are in Λ^n infinitely many incommensurable discrete arithmetic groups of motions with bounded fundamental domain[(2)], while for sufficiently large n there are no discrete reflection groups at all with these properties. (This is proved in the present article.)

The construction of arbitrary discrete groups of motions of Λ^n reduces in principle to the construction of their fundamental polytopes, together with generators for the group of motions that match pairwise with one another the $(n-1)$-dimensional faces of the fundamental polytope. Poincaré [52] proved that all finitely generated Fuchsian groups, that is, discrete groups of

[(1)]The number of degrees of freedom of a simple convex m-gon in Λ^n is $mn-n(n+1)/2$. (The subtrahend is the dimension of the group of motions.)

[(2)]For example, the groups $O'(f_d, A_d)$ (see the notation under **6** of this Introduction), when d is a squarefree integer $\neq 1$, A_d is the ring of integers of the quadratic field $\mathbf{Q}(\sqrt{d})$, and $f_d(x) = -\sqrt{d}x_0^2+x_1^2+ \dots +x_n^2$.

motions of the Lobachevskii plane, can be obtained by this method. In his memoir [53] on Kleinian groups he outlined a similar program for three-dimensional Lobachevskii space.

Poincaré's method was rigorously justified and generalized in various directions by A.D. Aleksandrov [1], Maskit [43], Seifert [24], and other mathematicians. However, in Lobachevskii spaces of dimension at least 3 it has been applied successfully only to groups generated by reflections and their subgroups of finite index (see, for example, [5], [11], [67], [68]). For groups generated by reflections it gives precisely the reduction to Coxeter polytopes of which we talked under 1. As for the subgroups of finite index, once a fundamental polytope of the whole group is known, the construction of their fundamental polytopes reduces to a purely combinatorial problem. In other cases the metric conditions imposed on the fundamental polytope are usually so complicated, and the combinatorial conditions so ambiguous, that the construction of the required polytopes seems an almost hopeless task(1).

A more substantive application of Poincaré's method to verify discreteness and to find a fundamental domain of a group of motions was guessed from other considerations, for example, within the framework of the ideas of Thurston. Thus, in [60] it is mentioned that Riley has used this approach to construct certain discrete groups of motions of Λ^3 having a bounded fundamental domain.

5. The first chapter of this paper is devoted to an algebraic description of acute-angled polytopes in Lobachevskii space. Namely, every such polytope $P \subset \Lambda^n$ is determined by its Gram matrix $G(P)$. In §2 the existence theorem for a polytope with given Gram matrix is proved. The combinatorial structure of P is described in terms of $G(P)$ in §3, while in §4 criteria are given for P to be bounded and to have finite volume. These results are due to the author ([10], [13], [14]). In essence they are theorems about certain special systems of linear inequalities in a pseudo-Euclidean vector space $E^{n,1}$. The proofs are based on the Perron-Frobenius theorem about matrices with non-negative entries.

In the second chapter the results obtained are applied to Coxeter polytopes. In this case the language of Coxeter schemes allows us to reduce matters to a calculation of determinants, and to visual work with graphs. Various examples are adduced. In particular, we treat from this point of view certain examples of bounded Coxeter polytopes in Λ^4 and Λ^5 constructed by Makarov [40]. We describe examples of bounded Coxeter polytopes of record dimension in Λ^6 and Λ^7, which are due to Bugaenko [71], and we discuss the classification of Coxeter simplicial prisms given by Kaplinskii [25].

(1)Recently, Mostov [46] has made brilliant use of Poincaré's method to construct several discrete groups of motions of complex hyperbolic space. However, the technical difficulties are so severe that from the beginning he was obliged to have recourse to a computer.

In §6 we show that in Lobachevskii space of dimension at least 62 there are no bounded Coxeter polytopes. The more refined analysis in [18] enables us to reduce the bound on the dimension to 30.

An essential part in the proof of this theorem is played by comparatively recent results on the combinatorics of convex polytopes with simplicial faces ([57], [22]), which by duality can be restated for simple polytopes. As Nikulin has observed, it follows from these results that the average complexity of the faces of given dimension of a simple polytope approaches that of the cube as the dimension of the polytope increases. In [49] he used this property as applied to three-dimensional faces. In the present paper and in [18] it is applied by two-dimensional faces, for which it means that: for a simple polytope of high dimension there must be in a certain sense many quadrilateral and triangular two-dimensional faces.

6. In this survey there are no results on problems of arithmeticity nor is there an investigation of *arithmetic discrete reflection groups* in Lobachevskii spaces. Therefore, we mention here the most important of these results, which are due mainly to the author and to Nikulin.

Let K be a totally real field of algebraic numbers, and A its ring of integers. A quadratic form

$$f(x)=\sum_{i,j=0}^{n} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}\in A)$$

is called *admissible* if it has signature $(n, 1)$, and for every non-identity embedding $\sigma : K \to \mathbf{R}$ the quadratic form

$$f^{\sigma}(x)=\sum_{i,j=0}^{n} a_{ij}^{\sigma}x_ix_j$$

is positive definite. It is known that in this case the group $O'(f, A)$ of integral linear transformations preserving f and mapping every connected component of the cone $\{x \in \mathbf{R}^{n+1} : f(x) < 0\}$ onto itself is a discrete group of motions of Lobachevskii space Λ^n (in the model described in § 1.3 of this paper). If $K \neq \mathbf{Q}$, or $K = \mathbf{Q}$ but f does not represent zero in $\mathbf{Q}$, then a fundamental domain in Λ^n of this discrete group is bounded [69]; in all other cases it is unbounded but has finite volume.

We denote by $O_r(f, A)$ the subgroup of $O'(f, A)$ generated by all reflections contained in it. The form f is said to be *reflective* if this subgroup is of finite index.

In [48] and [49] Nikulin proved that the number of reflective forms is finite up to proportionality and integral equivalence, for any fixed n and fixed degree of K. In [17] and [18] the author proved that reflective forms do not exist at all for $n \geqslant 30$. These results give hope for a classification of all reflective forms.

For $K = \mathbf{Q}$ it is proved in a sequence of papers [14]-[16], [20] (see also [44]) that a *unimodular* admissible quadratic form is reflective if and only if $k \leqslant 19$, and fundamental polytopes are found for the c.r.g. thus obtained. Similar work has been done for $K = \mathbf{Q}(\sqrt{5})$ by Bugaenko; in this case reflectivity holds for $n \leqslant 7$ [71].

We say that a c.r.g. Γ in Lobachevskii space Λ^n is *arithmetic* if it is contained (as a subgroup of finite index) in a group of the form $O'(f, A)$, where f is an (automatically reflective) admissible quadratic form over a totally real field K. In this situation we say that K is a *defining field* for Γ.

It is proved in [10] that this definition is compatible with the general definition of arithmetic discrete subgroups of semisimple Lie groups (see [55] for example), and a criterion is given for a c.r.g. to be arithmetic in terms of the Gram matrix of its fundamental polytope.

7. With every algebraic surface X of type K3 there is associated an intersection form on its lattice of algebraic cycles. This is an even integral quadratic form of signature $(1, n)$, where $n+1 \leqslant 20$ is the dimension of the lattice of algebraic cycles. Let f_X be its opposite form of signature $(n, 1)$. The group of automorphisms of X can be described in terms of f_X, up to a finite central extension ([54], [50]). In particular, it is finite if and only if the groups $O'(f_X, \mathbf{Z})$ contains a subgroup of finite index generated by reflections associated with integral vectors e such that $f_X(e) = 2$. We call them *2-reflections* and we call *2-reflective* the integral quadratic forms of signature $(n, 1)$ whose automorphism group contains a subgroup of finite index generated by 2-reflections.

It is clear that every 2-reflective form is reflective. The converse is false. For example, odd unimodular forms of signature $(n, 1)$ are reflective for $n \leqslant 19$, but there are 2-reflective forms only for $n \leqslant 15$, $n \neq 2, 10, 14$.

It is easy to see that an odd form is 2-reflective if and only if the even form canonically associated with it is 2-reflective. Therefore, in a study of 2-reflective forms we may restrict our attention to even forms.

In connection with the problem of classifying K3-surfaces with a finite automorphism group, Nikulin ([47], [50]) has listed all 2-reflective even integral quadratic forms of signature $(n, 1)$ for $n \geqslant 4$. It turns out that there are only finitely many of them and that they are all connected with K3-surfaces in the manner described. The largest possible value of n for them is 18.

Subsequently, the author succeeded in extending this classification to $n = 3$, and Nikulin did the same for $n = 2$. In both these dimensions the number of 2-reflective forms again turned out to be finite.

The theory of discrete reflection groups in Lobachevskii spaces enables us to compute effectively the automorphism group of a surface X of type K3 also when f_X is reflective, but not necessarily 2-reflective.

CHAPTER I

ACUTE-ANGLES POLYTOPES IN LOBACHEVSKII SPACES

§1. The Gram matrix of a convex polytope

1. Let X^n denote the n-dimensional Euclidean space E^n, the n-dimensional sphere S^n, or the n-dimensional Lobachevskii space Λ^n.

A *convex polytope* in X^n is a subset of the form

$$P = \bigcap_{i \in I} H_i^-, \tag{2}$$

where H_i^- is the closed half-space bounded by the hyperplane H_i, under the assumptions that

1) P contains a non-empty open subset of X^n;

2) every bounded subset of it intersects only finitely many hyperplanes H_i.

It may always be assumed that *none of the half-spaces H_i^- contains the intersection of all the others.* In what follows we assume this without special mention. Under this condition the half-spaces H_i^- are uniquely determined by P. We say that each of the H_i *bounds* the polytope P.

Throughout this paper we assume that I is a finite set. By 2), this holds automatically for *bounded* convex polytopes.

A convex polytope in Euclidean or Lobachevskii space is bounded if and only if it is the convex hull of finitely many points. In Lobachevskii space there are unbounded convex polytopes of *finite volume.* A convex polytope in Lobachevskii space of dimension greater than 1 has finite volume if and only if it is the convex hull of finitely many ordinary points or points at infinity.

A convex polytope is said to be *non-degenerate* if its bounding hyperplanes do not have a common ordinary point or (in the case of Lobachevskii space) a point at infinity, and there is no hyperplane orthogonal to all of them. Every convex polytope of finite volume in Euclidean or Lobachevskii space is non-degenerate.

2. Let P be a convex polytope in Euclidean space E^n, expressed in the form (2). For each $i \in I$ let e_i denote the unit vector orthogonal to H_i and starting at P. We define the *Gram matrix of P* as that of the system of vectors $\{e_i : i \in I\}$ and denote it by $G(P)$. This is a positive semidefinite symmetric matrix with 1's along the diagonal. For $i \neq j$ the entry g_{ij} is the negative of the cosine of the dihedral angle $H_i^- \cap H_j^-$ if H_i and H_j intersect, and -1 if they are parallel. If P is non-degenerate, then $rk\ G(P) = n$.

To define the Gram matrix of a spherical convex polytope we represent S^n by its natural embedding in Euclidean vector space E^{n+1}. Then every convex polytope $P \subset S^n$ is the intersection of S^n with a (uniquely determined) convex polyhedral cone $K(P) \subset E^{n+1}$. We define the *Gram matrix of P* as that of this cone (as a convex polytope in E^{n+1}) and again denote it by $G(P)$.

For $i \neq j$ the entry g_{ij} is the negative of the cosine of the spherical dihedral angle $H_i^- \cap H_j^-$. The matrix $G(P)$ determines P up to symmetry. The polytope P is non-degenerate if and only if $rk\ G(P) = n+1$.

The Gram matrix of a convex polytope in Lobachevskii space is defined similarly. To do this we first describe the model of Lobachevskii space that we shall use.

3. Let $E^{n,1}$ be a pseudo-Euclidean vector space of signature $(n, 1)$. We denote by C_+ and C_- the connected components of the open cone

$$C = \{x \in E^{n,1} : (x, x) < 0\}.$$

Let $O_{n,1}$ be the group of orthogonal transformations[1] of $E^{n,1}$ and $O'_{n,1}$ its subgroup of index 2 consisting of the transformations that map each connected component of C onto itself. Clearly, $O_{n,1} = O'_{n,1} \times \{1, -1\}$. Finally, let $\mathbf{R}_+$ be the group of positive numbers acting on $E^{n,1}$ by homothety.

With this notation, the n-dimensional Lobachevskii space can be identified with the quotient set $C_+/\mathbf{R}_+$ in such a way that motions are induced by linear transformations in $O'_{n,1}$.

In this model the space Λ^n is a domain in the manifold

$$PS^n = (E^{n+1} \setminus \{0\}) / \mathbf{R}_+.$$

The manifold PS^n is diffeomorphic to the sphere, but carries no natural spherical metric. Bearing in mind that it is a two-sheeted cover of projective space, we call it the *projective sphere* and apply the terminology of projective geometry in relation to it.

The closure $\overline{\Lambda}^n$ of Λ^n in PS^n is called its *completion*, and the points of the boundary $\partial\Lambda^n = \overline{\Lambda}^n \setminus \Lambda^n$ are its *points at infinity*. (Points of the projective sphere PS^n not in $C/\mathbf{R}_+$ are sometimes called *ideal points* of Λ^n.)

We denote by π the canonical map

$$\pi : E^{n,1} \to PS^n$$

(which is not defined at zero).

The restriction of the scalar product in $E^{n,1}$ to any $(k+1)$-dimensional subspace U has the signature $(k, 1)$, $(k, 0)$, or $(k+1, 0)$. Depending on which of these cases holds, we describe U as *hyperbolic*, *parabolic*, or *elliptic*, respectively. The same term is used for the corresponding k-dimensional plane $\pi(U)$ in the projective sphere. We remark that the orthogonal complement $U^\perp$ to a hyperbolic (or parabolic, or elliptic) subspace U is elliptic (or parabolic, or hyperbolic).

The planes of a Lobachevskii space in our model are the intersections with Λ^n of hyperbolic planes (of the same dimension) of the projective sphere PS^n.

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[1] By an orthogonal transformation of $E^{n,1}$ we mean a linear map preserving the scalar product of signature $(n, 1)$ in this space.

Every hyperplane of Λ^n can be represented in the form

$$H_e = \{\pi(x) : x \in C_+, \ (x, e) = 0\},$$

where e is a vector with positive scalar square. The closed half-spaces bounded by it are denoted by H_e^+ and H_e^-, so that

$$H_e^- = \{\pi(x): \ x \in C_+, \ (x, e) \leqslant 0\}.$$

The mutual disposition of hyperplanes H_e and H_f under the condition that $(e, e) = (f, f) = 1$ can be described as follows [70]. The hyperplanes H_e and H_f intersect (are parallel or ultraparallel) if and only if $|(e, f)| < 1$ (or $|(e, f)| = 1$, or $|(e, f)| > 1$, respectively). If they intersect, then the angle between them is determined by the formula

$$\cos \widehat{H_e H_f} = |(e, f)|;$$

if they are ultraparallel, then the distance between them is

$$\cosh \rho (H_e, H_f) = |(e, f)|.$$

The mutual disposition of half-spaces H_e^- and H_f^- is determined by the sign of (e, f). A negative sign indicates that one of the following three cases holds:

1) H_e and H_f intersect, and the dihedral angle $H_e^- \cap H_f^-$ is acute;
2) $H_e^- \supset H_f^+$ and $H_f^- \supset H_e^+$;
3) $H_e^- \cap H_f^- = \varnothing$.

4. Let P be a convex polytope in Lobachevskii space Λ^n, expressed in the form (2). For every $i \in I$ let e_i denote a vector in $E^{n,1}$ such that

$$(e_i, e_i) = 1, \quad H_i^- = H_{e_i}^-.$$

The system of vectors $\{e_i : i \in I\}$ determines P if it is known which of the connected components of C is used for C_+. In fact,

$$P = \pi (K \cap C_+) = \pi(K) \cap \Lambda^n,$$

where $K = K(P)$ is the convex polyhedral cone in $E^{n,1}$ given by

$$K = \{x \in E^{n,1} : (x, e_i) \leqslant 0 \ \text{ for all} \tag{3}$$

The *Gram matrix of the polytope* P is by definition that of the system of vectors $\{e_i : i \in I\}$ and is denoted by $G(P)$. By §1.3, its entries g_{ij} for $i \neq j$ have the following meaning:

1) if $|g_{ij}| < 1$, then g_{ij} is the negative of the cosine of the dihedral angle $H_i^- \cap H_j^-$;

2) if $g_{ij} \leqslant -1$, then g_{ij} is the negative of the hyperbolic cosine of the distance between H_i and H_j. (The case $g_{ij} \geqslant 1$ is impossible, since then $H_i^- \supset H_j^-$ or $H_j^- \supset H_i^-$.)

The polytope P is non-degenerate if and only if there is no vector in $E^{n,1}$ orthogonal to all the vectors e_i, $i \in I$; in other words, if and only if the e_i, $i \in I$ span $E^{n,1}$. This is equivalent to $rk\ G(P) = n = 1$.

If $G(P)$ is of rank $n+1$, then it determines the system of vectors $\{e_i : i \in I\}$ up to an orthogonal transformation of $E^{n,1}$. Since the group of motions of Λ^n is a subgroup of index 2 in $O_{n,1}$, there are up to a motion at most two non-degenerate convex polytopes with a given Gram matrix. This indeterminacy can occur only when the cone K intersects both connected components of C.

5. We say that a non-degenerate convex polytope $P \subset \Lambda^n$ is *decomposable* if it has a proper face F_0 of some dimension that is orthogonal to every hyperplane H_i not containing it. In this case the orthogonal projection onto the plane of the face F_0 determines a fibration of P into polyhedral cones with vertices at points of F_0, so that all vertices of P lie in F_0. Therefore, every convex polytope of finite volume is indecomposable.

A system of vectors in $E^{n,1}$ is said to be *decomposable* if it can be split into two mutually orthogonal subsystems.

It is easy to see that a polytope P is decomposable if and only if the system of vectors $\{e_i : i \in I\}$ is decomposable. The latter in its turn is equivalent to the condition that the Gram matrix $G(P)$ can be split into a direct sum of two principal submatrices (see §2.2).

§2. The existence theorem for an acute-angled polytope with given Gram matrix

1. In what follows we are interested in only those convex polytopes P that satisfy the following condition: if two hyperplanes H_i and H_j bounding the given polytope intersect, then the dihedral angle $H_i^- \cap H_j^-$ does not exceed $\pi/2$. In terms of $G(P)$ this means that

$$g_{ij} \leqslant 0 \quad \text{for} \quad i \neq j.$$

Allowing a certain *abus de language*, we call such polytopes *acute-angled.*

We remark that according to a result of Andreev [4], if the dihedral angles for all $(n-2)$-dimensional faces of P do not exceed $\pi/2$, then the hyperplanes of its non-adjacent $(n-1)$-dimensional faces do not intersect, consequently, it satisfies the condition stated above. However, we do not need this result.

2. We come now to some results on *symmetric matrices with non-positive entries off the diagonal*, among which there are, in particular, the Gram matrices of convex polytopes with acute dihedral angles.

A square matrix A is said to be the *direct sum* of the (square) matrices $A_1, A_2, \ldots, A_k$ if by some permutation of the rows and the same permutation of the columns it can be brought to the form

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{bmatrix}.$$

In this case we write

A matrix that cannot be represented as a direct sum of two matrices is said to be *indecomposable.* Every matrix can be represented uniquely as a direct sum of indecomposable matrices, its so-called (*indecomposable*) *components.*

Let $A = (a_{ij})$ be an indecomposable symmetric matrix with non-positive entries off the diagonal. We denote by $\lambda(A)$ the smallest eigenvalue of A, and by $\delta(A)$ the largest diagonal entry. Then A can be written in the form

$$A = \delta(A)E - B,$$

where B is an indecomposable symmetric matrix with non-positive entries. Applying the Perron-Frobenius theorem to B we find that

1) $\lambda(A)$ is a simple eigenvalue of A;

2) the corresponding eigenvector has positive coordinates.

The properties of A depend in the first place on the sign of $\lambda(A)$. If $\lambda(A) > 0$, then A is positive definite, hence is non-degenerate. In this case all the entries of A^{-1} are positive [21]. If $\lambda(A) = 0$, A is positive semidefinite and degenerate. However, since 0 is a simple eigenvalue, the rank of A in this case is smaller by 1 than its order. The coefficients of a linear dependence relation between the rows of A are none other than the coordinates of an eigenvector corresponding to the zero eigenvalue, consequently, are positive. Thus, every principal proper submatrix of A is positive definite.

A symmetric matrix with non-positive entries off the diagonal whose components are all positive definite and degenerate is called *parabolic.*

3. From the properties of positive semidefinite matrices with non-positive entries off the diagonal mentioned in 2 one obtains the following description of acute-angled polytopes in Euclidean space and on the sphere [28].

Let A be a positive semidefinite matrix of rank n with 1's on the diagonal and non-positive entries off it.

1) If A is non-degenerate, then it is the Gram matrix of a simplicial cone in E^n defined up to a motion, and at the same time it is the Gram matrix of a simplex in S^{n-1} that is likewise defined up to a motion;

2) if A is indecomposable and degenerate, then it is the Gram matrix of a simplex in E^n, which is defined up to similarity.

Every acute-angled polytope in Euclidean space is the direct product of a number of simplexes (corresponding to the degenerate components of the Gram matrix), a simplicial cone of some dimension (corresponding to the sum of its non-degenerate components), and a Euclidean space of a certain dimension. In particular, every bounded acute-angled polytope in Euclidean space is a direct sum of simplexes.

Every non-degenerate spherical acute-angled polytope is a simplex.

The Gram matrix of a non-degenerate acute-angled polytope in Lobachevskii space Λ^n is symmetric and of signature $(n, 1)$ with 1's along the diagonal and non-positive entries off it. The order of such a matrix can be arbitrarily large (for fixed n). Therefore, we cannot hope for a relatively simple description of the possible combinatorial types of acute-angled polytopes in Lobachevskii space. However, there is an existence and uniqueness theorem for a polytope with given Gram matrix, but its proof (in essence, the existence proof) requires a much more subtle analysis of linear inequalities than in the Euclidean case. This theorem is the main result of this section. Its proof will be given in the following sub-sections.

4. ***Proposition*** 2.1. *Let $\{e_i : i \in I\}$ be an indecomposable finite system of vectors spanning $E^{n,1}$ and such that $(e_i, e_j) \leqslant 0$ for $i \neq j$. Then the cone K defined by the linear inequalities $(x, e_i) \leqslant 0$ $(i \in I)$ contains a non-empty open subset of one of the connected components of C and does not contain non-zero vectors from the closure of the other component.*

Proof. Let $G = (g_{ij})$ be the Gram matrix of the system of vectors $\{e_i : i \in I\}$. By hypothesis, it is indecomposable. Let $\lambda = \lambda(G) < 0$ be the least eigenvalue and $c_i > 0$, $i \in I$, the coordinates of a corresponding eigenvector. We consider the vector

$$v = \sum_j c_j e_j.$$

We have:

$$(v, e_i) = \sum_j g_{ij} c_j = \lambda c_i < 0 \quad \text{for all } i \in I.$$

This means that v is an interior point of K. At the same time,

$$(v, v) = \sum_i c_i (v, e_i) < 0,$$

so that $v \in C$. Suppose, to be definite, that $v \in C_+$; then $(v, x) > 0$ for $x \in \bar{C}_- \setminus \{0\}$. However, for $x \in K$

$$(v, x) = \sum_i c_i (e_i, x) \leqslant 0.$$

Thus, $K \cap \bar{C}_- = \{0\}$.

5. ***Proposition*** 2.2. *Let $\{e_i : i \in I\}$ be a finite system of vectors in $E^{n,1}$ such that*

$$(e_i, e_i) > 0, \quad (e_i, e_j) \leqslant 0 \quad \textit{for} \quad i \neq j.$$

Suppose that the cone K defined by the inequalities $(x, e_i) \leqslant 0$ $(i \in I)$ contains a non-empty open subset of C_+. Then the convex polytope $P = \pi(K \cap C_+)$ in Λ^n is bounded by the hyperplanes $H_i = H_{e_i}$ $(i \in I)$, and for every $i \in I$ the orthogonal projection of P on H_i is contained in P.

Proof. For $j \in I$ we denote by E_j the orthogonal complement to e_j in $E^{n,1}$. The orthogonal projection x' of the vector $x \in E^{n,1}$ on E_j is given by the formula

$$x' = x - \frac{(x, e_j)}{(e_j, e_j)} e_j.$$

If $x \in K$, then for $i \neq j$

$$(x', e_i) = (x, e_i) - \frac{(x, e_j)(e_j, e_i)}{(e_j, e_j)} \leqslant 0,$$

hence, $x' \in K$.

Suppose now that $p = \pi(x) \in P$. The orthogonal projection of p on the hyperplane $H_j = \pi(E_j \cap C_+)$ is the point $p' = \pi(x')$, where x' is the orthogonal projection of x on E_j. By what was said above, $p' \in P$. Thus, the orthogonal projection of P on H_j lies in P. Since it contains a non-empty open subset of H_j, this hyperplane bounds P.

We are now in a position to prove the main theorem of this section.

***Theorem* 2.1.** *Let $G = (g_{ij})$ be an indecomposable symmetric matrix of signature $(n, 1)$ with 1's along the diagonal and non-positive entries off it. Then there is a convex polytope P in Λ^n whose Gram matrix is G. The polytope P is uniquely determined up to a motion in Λ^n.*

Proof. We assume that the rows and columns of G are indexed by the elements of some set I. Let $\{e_i : i \in I\}$ be a system of vectors in $E^{n,1}$ whose Gram matrix is G, and let K be the convex polyhedral cone defined by the inequalities $(x, e_i) \leqslant 0$ $(i \in I)$. By Proposition 2.1, K contains a non-empty open subset of one of the connected components of C, and does not intersect the other component. We may assume that $K \cap C_+ \neq \emptyset$. Then $P = \pi(K \cap C_+)$ is a convex polytope in Λ^n. By Proposition 2.2, each of the hyperplanes $H_i = H_{e_i}$, $i \in I$, bounds P. It follows from this that the Gram matrix is G.

Since K intersects only one of the connected components of C, P is uniquely determined up to a motion in Λ^n (see § 1.4).

The fact that the Gram matrix of a convex polytope is indecomposable is equivalent to the polytope itself being indecomposable (see § 1.5). Therefore, the theorem just proved gives an algebraic description of all indecomposable acute-angled polytopes in n-dimensional Lobachevskii space.

§3. Determination of the combinatorial structure of an acute-angled polytope from its Gram matrix

1. Let $P \subset X^n$ be a convex polytope presented in the form (2). (We recall our assumption that none of the half-spaces H_i^- contains the intersection of the rest.) For every face F of P we set

$$\iota(F) = \{i \in I\colon F \subset H_i\}.$$

We denote the set of all subsets of the form $\iota(F)$ (including $\iota(P) = \varnothing$) of I by $\mathcal{F}(P)$ and call it the *complex of the polytope P.* Under inclusion $\mathcal{F}(P)$ is a partially ordered set anti-isomorphic to the partially ordered set of faces of P. The codimension of F is the "height" of the element $\iota(F)$ in $\mathcal{F}(P)$, that is, the greatest length of an ascending chain that can be formed from elements less than $\iota(F)$. Thus, the complex of a polytope carries within it the whole information on its combinatorial structure.

We give two examples that are needed in what follows. The complex of an n-dimensional simplex in Euclidean space, and also of a bounded simplex in Lobachevskii(1) space, consists of all proper subsets of $I = \{1, 2, \ldots, n+1\}$. The complex of a direct product of a k-dimensional and an l-dimensional simplex in Euclidean space E^n $(n = k+l)$ under a suitable numbering of the $(n-1)$-dimensional faces consists of all subsets of $I = \{1, 2, \ldots, n+2\}$ not containing any one of the subsets of $I_1 = \{1, 2, \ldots k+1\}$ and of $I_2 = \{k+2, \ldots, n+2\}$.

It turns out that if P is an acute-angled polytope in Lobachevskii space (see §2.1), then the complex $\mathcal{F}(P)$ has a simple description in terms of the Gram matrix $G(P) = G$.

For each subset $J \subset I$ we use the following notation:

E_J is the linear span of the vectors $\{e_j\colon j \in J\}$,

$E^J = E_J^\perp$,

$P^J = P \cap \pi(E^J) = P \cap (\bigcap_{j \in J} H_j)$,

G_J is the principal submatrix of G formed from the rows and colums whose indices belong to J.

If $J \in \mathcal{F}(P)$, then $P^J = F$ is that face of P for which $\iota(F) = J$.

***Theorem* 3.1.** *Let $P = \bigcap_{i \in I} H_i^- \subset \Lambda^n$ be an acute-angled polytope and $G = G(P)$ its Gram matrix. A subset $J \subset I$ lies in $\mathcal{F}(P)$ if and only if the matrix G_J is positive definite, and in that case*

$$\operatorname{codim} P^J = |J|,$$

where $|J|$ is the number of elements in J.

(1)In general, by a *simplex* in Λ^n we mean the convex hull of $n+1$ ordinary points or points at infinity not lying in a single hyperplane. Thus, a simplex always has finite volume, but may be unbounded. In the latter case its complex does not contain certain n-element subsets of $I = \{1, 2, \ldots, n+1\}$.

Proof. Suppose that $J = \iota(F)$, where F is a face of P. Since $\pi(E^J) \cap \Lambda^n$ is the plane of F, it follows that E^J is a hyperbolic subspace of $E^{n,1}$, while $E_J = (E^J)^{\perp}$ is elliptic. Consequently, G_J, which is the Gram matrix of the system of vectors $\{e_j: j \in J\}$, is positive semidefinite. If it were degenerate, there would be a non-trivial linear dependence with non negative coefficients between its rows, and hence also between the vectors e_j (see §2.2). In that case certain of the inequalities (3) on the cone K would have to become equalities, which is impossible, since this would mean that P is contained in a hyperplane. Thus, G_J is positive definite and the vectors e_j, $j \in J$, form a basis for E_J; in particular codim $F = \dim E_J = |J|$.

The reverse implication will be proved in the following section, but right now we mention an important consequence of the part of the theorem already proved.

A convex polytope is said to be *simple* if each of its faces of codimension m is contained in exactly m faces of codimension 1.

Corollary. *Every acute-angled polytope is simple.*

2. To complete the proof of Theorem 3.1 we use the following proposition, which can be regarded as a generalization of Proposition 2.2.

Proposition 3.1. *Let $P = \bigcap_{i\in I} H_i^- \subset \Lambda^n$ be an acute-angled polytope and $G = G(P)$ its Gram matrix. Further, let $J \subset I$ be a subset such that the matrix G_J is positive definite. We set $|J| = m$. Then P^J is a face of codimension m of P, and the orthogonal projection of P on the plane of this face is contained in P (and so in P^J).*

Proof. Since G_J is a positive definite, the vectors e_j, $j \in J$, form a basis of an m-dimensional elliptic subspace. Consequently, E^J is a hyperbolic subspace of codimension m, and $\Pi^J = \pi(E^J \cap C_+)$ is a plane of codimension m in Λ^n.

The orthogonal projection x' of a vector $x \in E^{n,1}$ on the subspace E^J is found by the formula

$$x' = x - \sum_{j,\, k\in J} h_{jk}(x, e_j)\, e_k,$$

where the h_{jk} $(j, k \in J)$ are the entries of G_J^{-1}. Since G_J is a positive definite symmetric matrix with non-positive entries off the diagonal, we see that $h_{jk} \geqslant 0$ (see §2.2). Thus, if $x \in K$, then for $i \notin J$

$$(x', e_i) = (x, e_i) - \sum_{j,\, k\in J} h_{jk} g_{ki}(x, e_j) \leqslant 0,$$

hence $x' \in K$.

Suppose now that $p = \pi(x) \in P$. The orthogonal projection of p to the plane Π^J is $p' = \pi(x')$, where x' is the orthogonal projection of x to the subspace E^J. By what was proved above, $x' \in K$, and this means that $p' \in P$.

Thus, the orthogonal projection of P on Π^J lies in P. Since it contains a non-empty open subset of Π^J, we see that $P^J = \Pi^J \cap P$ is a face of codimension m of P. This completes the proof of the proposition.

Combining this proposition with the part of Theorem 3.1 already established, we obtain the following pretty geometric theorem, which also has a simple geometric proof.

Corollary. *The orthogonal projection of an acute-angled polytope in Lobachevskii space to the plane of any of its faces lies within that face.*

(A similar theorem holds in Euclidean space and on the sphere.)

Conclusion of the proof of Theorem 3.1. Let $J \subset I$ be a subset such that G_J is positive definite. We set $|J| = m$. By Proposition 3.1, $P^J = F$ is a face of codimension m of P. By definition, $\iota(F) \supset J$, but it was proved above that $|\iota(F)| = \operatorname{codim} F$; consequently, $\iota(F) = J$.

3. By a *vertex at infinity* of a convex polytope $P \subset \Lambda^n$ we mean a point at infinity of Λ^n contained in the closure of P and having the following property: the intersection of P with any sufficiently small horosphere with centre at q is a bounded subset (a convex polytope) of the horosphere regarded as an $(n-1)$-dimensional Euclidean space.

In many circumstances, vertices at infinity play the same role as ordinary vertices. For unbounded convex polytopes they are naturally included in the picture of the combinatorial structure.

We give a more convenient characterization of vertices at infinity.

Let $P \subset \Lambda^n$ be a converse polytope presented in the form (2). For every point at infinity $q = \pi(u) \in \overline{P}$ we set

$$\iota(q) = \{i \in I\colon\ q \in \overline{H}_i\} = \{i \in I\colon\ (u,\ e_i) = 0\}.$$

We denote by $\widetilde{E}_q$ the orthogonal complement to the vector u in $E^{n,1}$. This is a parabolic subspace, and the quotient space $E_q = \widetilde{E}_q/\mathbf{R}$ is $(n-1)$-dimensional Euclidean. We denote by $\overline{e}_j$ $(j \in \iota(q))$ the image of e_j under the canonical map $\widetilde{E}_q \to E_q$.

Lemma 3.1. *A point at infinity $q \in \overline{P}$ is a vertex at infinity of P if and only if among the $\overline{e}_j$, $j \in \iota(q)$, there are vectors spanning E_q and connected by a linear dependence relation with positive coefficients.*

Proof. In a small neighbourhood of q in PS^n the polytope P is the intersection with Λ^n of a polyhedral convex cone $\pi(K_q)$ (with vertex at q), where

$$K_q = \{x \in E^{n,1}\colon\ (x,\ e_j) \leqslant 0, \text{ for all } j \in \iota(q)\}.$$

The horosphere S_q of Λ^n with centre at q in the geometry of the projective sphere PS^n is an ellipsoid punctured at q and touching the boundary of Λ^n at this point. For the intersection of $\pi(K_q)$ with S_q to be a bounded subset of this horosphere it is necessary and sufficient that the intersection of $\pi(K_q)$ with the hyperplane $\pi(\widetilde{E}_q)$ that touches the boundary of Λ^n at q (and also the horosphere S_q), is the single point q. In terms of $E^{n,1}$ this means that $K_q \cap \widetilde{E}_q = \mathbf{R}u$.

The latter is equivalent to the condition that the system of linear inequalities

$$(x, \bar{e}_j) \leqslant 0, \quad j \in \iota(q),$$

has only the zero solution in E_q. By a standard theorem on linear inequalities (see, for example, [64]) this holds if and only if the $\bar{e}_j$ satisfy the condition of the lemma.

4. We denote by $\bar{\mathcal{F}}(P)$ the collection of subsets of I obtained by adding to $\mathcal{F}(P)$ all subsets of the form $\iota(q)$, where q is a vertex at infinity of P, and we call it the *extended complex* of P.

For acute-angled polytopes the extended complex can also be described simply in terms of the Gram matrix.

***Theorem* 3.2.** *Let $P = \bigcap_{i \in I} H_i^- \subset \Lambda^n$ be a polytope with acute dihedral angles, and $G = G(P)$ its Gram matrix. A subset $J \subset I$ is of the form $\iota(q)$, where q is a vertex at infinity of P, if and only if G_J is a parabolic matrix* (*see* §2.2) *of rank $n-1$.*

Proof. Suppose that $J = \iota(q)$, where q is a vertex at infinity. Then G_J is the Gram matrix of the system of vectors $\{\bar{e}_j \colon j \in \iota(q)\}$ of Euclidean space E_q (see §2.3), consequently is positive semidefinite. By Lemma 3.1, this system of vectors spans E_q. Thus, $rk\ G_J = n-1$. The decomposition of G_J into a direct sum of indecomposable matrices corresponds to the decomposition of the system of vectors $\{\bar{e}_j \colon j \in \iota(q)\}$ into mutually orthogonal subsystems, each of which is either linearly independent or connected by a unique relation with positive coefficients. If at least one of these systems is linearly independent, then the conditions of Lemma 3.1 cannot hold. Therefore, all the components of G_J are degenerate. This completes the "only if" assertion of the theorem.

5. The "if" assertion will be deduced from the following proposition, which is of independent interest.

***Proposition* 3.2.** *Under the conditions of Theorem 3.2, let $J \subset I$ be a subset such that G_J is parabolic. Let $c_j > 0$ ($j \in I$) be the coefficients of a linear dependence relation between the rows of G_J. Then the vector*

$$u = u(J) = \sum_{j \in J} c_j e_j$$

is non-zero, isotropic, and orthogonal to all the vectors e_j, $j \in J$, and

$$\pi(u) \in \bar{P}.$$

Proof. That u is orthogonal to the vectors e_j, $j \in J$, and that it is isotropic follow immediately from its definition. It is also clear that $(u, e_i) \leqslant 0$ for $i \notin J$. Thus, $u \in K$.

The vector u cannot be zero, since then the inequalities $(x, e_j) \leqslant 0$ $(j \in J)$ could be satisfied simultaneously only as equations, and P would be contained in a hyperplane. Nor can it lie in $\overline{C}_-$, since then we would have $(x, u) > 0$ for any $x \in C_+$, while $(x, u) = \sum_{j \in J} c_j(x, e_j) \leqslant 0$ for $\pi(x) \in P$. Thus, $\pi(u) \in \overline{P}$.

Conclusion of the proof of Theorem 3.2. Suppose that the matrix G_J satisfies the conditions of the theorem, and let $u = u(J)$ be a vector constructed as in Proposition 3.2. Then $q = \pi(u) \in \overline{P}$, so that $\iota(q) \supset J$.

In the notation of Lemma 3.1, the Gram matrix of the system of vectors $\{\overline{e}_j \colon j \in J\}$ is G_J. It follows from the properties of G_J that its rows, and hence also the $\overline{e}_j$, $j \in J$, are connected by a linear dependence relation with positive coefficients, and that the rank of the system of vectors $\{\overline{e}_j \colon j \in J\}$ is $n-1$. Thus, q satisfies the conditions of Lemma 3.1, consequently, it is a vertex at infinity of P.

We claim that $\iota(q) = J$. Since the vectors e_i with $i \in \iota(q)$ lie in $\overline{E}_q$, the matrix $G_{\iota(q)}$ is positive semidefinite and $\mathrm{rk} G_{\iota(q)} \leqslant n-1$. Every component of G_J is contained in some component of $G_{\iota(q)}$, but since it is degenerate, it must in fact be a component of $G_{\iota(q)}$. Thus, every component of $G_{\iota(q)}$ must be contained in G_J, since otherwise $rk\ G_J \leqslant n-1$. Thus, $\iota(q) = J$.

§4. Criteria for an acute-angled polytope to be bounded and to have finite volume

1. Let P be a non-degenerate convex polytope in Λ^n, expressed in the form (2). Generally speaking, it is hard to establish whether P is bounded, and if it is not bounded whether it has finite volume.

For a theoretical investigation of this problem it is useful to consider the "ideal" convex polytope

$$\hat{P} = \pi(K) \subset PS^n,$$

where K is the convex polyhedral cone given by the inequalities (3). We call it the *continuation* of P. It is clear that

$$P = \hat{P} \cap \Lambda^n. \tag{4}$$

Since the vectors e_i, $i \in I$, span $E^{n,1}$, the cone K is strongly convex (that is, it does not contain any one-dimensional subspaces), consequently $\hat{P}$ is the convex hull of its vertex set and can be constructed combinatorially as a bounded convex polytope in n-dimensional Euclidean space.

The polytope P is *bounded* (*has finite volume,* respectively) if and only if $\hat{P} = P$ ($\hat{P} = \overline{P}$, respectively).

The hyperplanes of the projective sphere bounding $\hat{P}$ are none other than the continuations of the hyperplanes in Lobachevskii space bounding P, and they are indexed by the same set I.

By (4), every face of P is part of a face of $\hat{P}$ of the same dimension. Moreover, every vertex at infinity of P is a vertex of P. Therefore,

$$\mathcal{F}(P) \subset \bar{\mathcal{F}}(P) \subset \mathcal{F}(\hat{P}).$$

A subset $J \in \mathcal{F}(\hat{P})$ lies in $\mathcal{F}(P)$ if and only if $P^J \neq \emptyset$. It follows from this that the complex $\mathcal{F}(P)$ is a *segment of* $\mathcal{F}(\hat{P})$ in the sense that whenever $J_1 \in \mathcal{F}(P)$, $J_2 \in \mathcal{F}(\hat{P})$ and $J_2 \subset J_1$, then $J_2 \in \mathcal{F}(P)$. It follows from the definition of a vertex at infinity that the extended complex $\bar{\mathcal{F}}(P)$ is a segment of $\mathcal{F}(\hat{P})$.

Since $\hat{P}$ is the convex hull if its vertex set, $\hat{P} = P$ (or $\hat{P} = \bar{P}$) is equivalent to $\mathcal{F}(\hat{P}) = \mathcal{F}(P)$ (or to $\mathcal{F}(\hat{P}) = \bar{\mathcal{F}}(P)$).

In the next section we indicate some necessary and sufficient conditions for a segment of the complex of a bounded convex polytope to be the whole complex. Applying these conditions to the segments $\mathcal{F}(P)$ and $\bar{\mathcal{F}}(P)$ of $\mathcal{F}(\hat{P})$, we obtain criteria for P to be bounded and to have finite volume.

2. Let Q be a bounded convex polytope in n-dimensional Euclidean or Lobachevskii space.

To the vertices of Q there correspond elements of height n of $\mathcal{F}(Q)$, and to the edges elements of height $n-1$ (see §3.1). Since every face of Q contains some vertex, the elements of height n are all the maximal elements of $\mathcal{F}(Q)$. Since every edge of Q contains exactly two vertices, every element of height $n-1$ of $\mathcal{F}(Q)$ is majorized by exactly two maximal elements.

We say that maximal elements of $\mathcal{F}(Q)$ majorizing a single element of height $n-1$ are adjacent. Since every pair of vertices of Q can be joined by a polygonal path of edges, every pair of maximal elements of $\mathcal{F}(Q)$ can be joined by a sequence of maximal elements in which every term is adjacent to the preceding one.

Proposition 4.1. *Let Q be a bounded convex polytope in n-dimensional Euclidean or Lobachevskii space. A segment $\mathfrak{J}$ of the complex $\mathcal{F}(Q)$ is the whole complex if and only if one of the following conditions is satisfied*:

1) *$\mathfrak{J}$ contains at least one element of height n, and for every such element and every element L of height $n-1$ majorized by it $\mathfrak{J}$ contains another element of height n majorizing L*;

2) *$\mathfrak{J}$ is isomorphic as a partially ordered set to the complex of a certain bounded convex polytope in n-dimensional* (*Euclidean or Lobachevskii*) *space.*

Proof. The necessity of both conditions is clear. Let us prove their sufficiency. If 1) is satisfied, then by what was said above $\mathfrak{J}$ contains two maximal elements of $\mathcal{F}(Q)$, consequently, coincides with $\mathcal{F}(Q)$. If 2) holds, then so does 1), hence $\mathcal{J} = \mathcal{F}(Q)$.

3. Applying Proposition 4.1 to the continuation of a convex polytope in Lobachevskii space, we obtain the following result.

***Proposition* 4.2.** *Let P be a non-degenerate convex polytope in n-dimensional Lobachevskii space. Then P is bounded (has finite volume, respectively) if and only if any one of the following conditions holds:*

1) P contains at least one vertex (ordinary or at infinity) and for every vertex (ordinary or at infinity) and every edge emanating from it there is another vertex (ordinary or at infinity) of P on that edge;

2) the complex $\mathcal{F}(P)$ (or $\overline{\mathcal{F}}(P)$) is isomorphic as a partially ordered set to the complex of some bounded n-dimensional convex polytope.

In particular, boundedness (or finiteness of the volume) of P depends only on $\mathcal{F}(P)$ (or $\overline{\mathcal{F}}(P)$).

The first criterion is of an algorithmic nature, but it is not effective in practice. The second can be applied successfully in cases when the combinatorial structure of P can be guessed.

4. There is another approach to investigating whether a polytope P is bounded or has finite volume.

For every subset $J \subset I$ we set

$$K^J = K \cap E^J, \quad \hat{P}^J = \pi(K^J)$$

(see the notation in §3.1).

Let us assume that P is not bounded. Then $\mathcal{F}(\hat{P}) \neq \mathcal{F}(P)$. For every $J \in \mathcal{F}(\hat{P}) \setminus \mathcal{F}(P)$,

$$\hat{P}^J \neq \varnothing, \quad P^J = \varnothing. \tag{5}$$

Let J be minimal among the subsets of I satisfying (5). Then it is also minimal among the subsets for which $P^J = \varnothing$.

Every minimal subset $J \subset I$ for which $P^J = \varnothing$ is called *critical*.

The preceding argument shows that if P is not bounded there exists a critical subset $J \subset I$ such that $\hat{P}^J \neq \varnothing$. If it is bounded, then $\hat{P}^J = P^J = \varnothing$ for every critical subset J.

Similarly, if P has finite volume, then $\mathcal{F}(\hat{P}) \neq \overline{\mathcal{F}}(P)$, and for every $J \in \mathcal{F}(\hat{P}) \setminus \overline{\mathcal{F}}(P)$:

(6) $\hat{P}^J \neq \varnothing$, $\hat{P}^J$ is not a vertex at infinity, and $P^J = \varnothing$. Every minimal subset $J \subset I$ satisfying (6) is critical. Therefore, there is a critical subset J for which $\hat{P}^J \neq \varnothing$ and $\hat{P}^J$ is not a vertex at infinity of P. If P has finite volume, then for every critical subset J either $\hat{P}^J = P^J = \varnothing$ or $\hat{P}^J$ is a vertex at infinity.

We have thus proved the following proposition.

***Proposition* 4.3.** *Let P be a non-degenerate convex polytope in n-dimensional Lobachevskii space. Then P is bounded (or has finite volume) if and only if $\hat{P}^J = \emptyset$ ($\hat{P}^J = \emptyset$ or $\hat{P}^J$ is a vertex at infinity of P, respectively) for every critical subset $J \subset I$.*

5. When P is an *acute-angled* polytope, Proposition 4.3 can be used to obtain more effective criteria for boundedness and for the volume to be finite.

We note first of all that in this case the property of a subset $J \subset I$ to be critical depends only on the corresponding principal submatrix G_J of the Gram matrix $G = G(P)$. Namely, Theorem 3.1 shows that $P^J \neq \emptyset$ if and only if G_J is positive definite. Therefore, a subset J is critical if and only if G_J is a minimal positive indefinite submatrix of G.

A symmetric matrix A with non-positive entries off the diagonal is called *critical* if it is not positive definite but every proper principal submatrix of it is positive definite. A critical matrix is automatically indecomposable, since otherwise every component of it would be positive definite, hence, so would the matrix itself be. A critical matrix can be either degenerate and positive semidefinite and degenerate, or non-degenerate and indefinite with negative index of inertia 1.

By what was said above, a subset $J \subset I$ is critical if and only if G_J is critical.

For every subset $J \subset I$ we set

$$Z(J) = \{i \in I\colon\ g_{ij} = 0 \text{ for all } j \in J\}, \quad N(J) = J \cup Z(J).$$

According to this definition, $E_{Z(J)}$ is an orthogonal subspace of E_J.

If J is a critical subset, then G_J is indecomposable and is either parabolic or non-degenerate and indefinite. In the first case E_J is a parabolic subspace; consequently, $E_{Z(J)}$ is parabolic or elliptic and $G_{Z(J)}$ is positive semidefinite. In the second case E_J is hyperbolic; consequently, $E_{Z(J)}$ is elliptic. Since the vectors e_i, $i \in Z(J)$, cannot be connected by a non-trivial linear dependence relation with non-negative coefficients, $G_{Z(J)}$ is positive definite in this case.

***Lemma* 4.1.** *Suppose that $J \subset I$, $L \subset Z(J)$, and the matrix G_L is positive definite. Then $\hat{P}^J = \emptyset$ whenever $\hat{P}^{J \cup L} = \emptyset$.*

Proof. An induction on the number of components of G_L reduces the proof to the case when G_L is indecomposable. The condition $\hat{P}^{J \cup L} = \emptyset$ means that between the vectors e_i, $i \in I$ there is a linear dependence relation $\sum\limits_i c_i e_i = 0$ in which $c_i > 0$ if $i \notin J \cup L$. Then for each $l \in L$:

$$(7) \qquad \sum_{k \in L} c_k g_{kl} = -\sum_{i \notin J \cup L} c_i g_{il} \geqslant 0,$$

where the inequality is strict for at least one l (otherwise G_L would be a direct summand of G).

By hypothesis, G_L is positive definite. Let us assume that it is indecomposable. Then all entries of the inverse matrix G_L^{-1} are positive, and from (7) it follows that $c_k > 0$ for $k \in L$. Thus, $c_i > 0$ for all $i \notin J$, and this means that $\hat{P}^J = \emptyset$. This completes the proof of the lemma.

***Theorem* 4.1.** *Let $P = \bigcap_{i \in I} H_i^- \subset \Lambda^n$ be an indecomposable acute-angled polytope and $G = G(P)$ its Gram matrix. Then P is bounded if and only if G_J is indefinite for every critical subset of G_J and*

$$\hat{P}^{N(J)} = \emptyset. \tag{8}$$

Moreover, P has finite volume if and only if for every critical subset $J \subset I$ either the preceding conditions hold or else $G_{N(J)} = G_J \oplus G_{Z(J)}$ is a parabolic matrix of rank $n-1$. (In this case $\hat{P}^{N(J)}$ is a vertex at infinity of P.)

Proof. We consider the conditions of Proposition 4.3 for some critical subset J. If G_J is positive semidefinite, then $\hat{P}^J \ni \pi(u)$, where $u = u(J)$ is the isotropic vector constructed as in Proposition 3.2. Of the two conditions of Proposition 4.3, only the second can be satisfied in this case: "$\hat{P}^J$ is a vertex at infinity". By Theorem 3.2, the latter holds if and only if G_J is a direct summand of the parabolic matrix G_L $(L \supset J)$ of rank $n-1$. Here automatically $L = N(J)$, since $G_{N(J)}$ is positive semidefinite and G_L cannot be a direct summand of it.

If G_J is indefinite, then $\hat{P}^J$ is contained in the elliptic plane $\pi(E^J)$ on the projective sphere, therefore cannot be a vertex at infinity of P. Consequently, of the two conditions of Proposition 4.3 only the first can hold in this case: "$\hat{P}^J = \emptyset$", whereas by Lemma 4.1 this is equivalent to "$\hat{P}^{N(J)} = \emptyset$". This completes the proof of the theorem.

Condition (8) seems hard to verify. However, it is necessarily satisfied if

$$|N(J)| = n + 1, \tag{9}$$

since then the vectors e_i, $i \in N(J)$ form a basis of $E^{n,1}$. In general, to check the condition it is enough to analyse only part of the complex $\mathscr{F}(\hat{P})$, which is the smaller, the larger $|N(J)|$.

In fact, with every subset $L \in \mathscr{F}(P)$ we associate the "relative complexes" $\mathscr{F}^L(P)$, $\overline{\mathscr{F}}^L(P)$, and $\mathscr{F}^L(\hat{P})$, the parts of $\mathscr{F}(P)$, $\overline{\mathscr{F}}(P)$, and $\mathscr{F}(\hat{P})$ consisting of subsets containing L. The complexes $\mathscr{F}^L(P)$ and $\mathscr{F}^L(\hat{P})$ are naturally isomorphic to $\mathscr{F}(P^L)$ and $\mathscr{F}(\hat{P}^L)$. (However, generally speaking $\overline{\mathscr{F}}^L(P)$ is not isomorphic to $\overline{\mathscr{F}}(P^L)$, since a vertex at infinity in the face P^L is not necessarily a vertex at infinity of P.) The complexes $\mathscr{F}^L(P)$ and $\overline{\mathscr{F}}^L(P)$ are segments of $\mathscr{F}^L(\hat{P})$. The condition $\mathscr{F}^L(\hat{P}) = \mathscr{F}^L(P)$ (or $\mathscr{F}^L(\hat{P}) = \overline{\mathscr{F}}^L(P)$) is equivalent to $\hat{P}^L \subset \Lambda^n$ (or $\hat{P}^L \subset \overline{\Lambda^n}$).

Suppose now that $J \subset I$ is a critical subset such that G_J is indefinite. We choose any $j \in J$ and set $L = N(J) \setminus \{j\}$. Then $L \in \mathcal{F}(P)$, and dim $P^L = n+1-|N(J)|$. If P is bounded (or has finite volume), then $\hat{P}^L \subset \Lambda^n$ (or $\hat{P}^L \subset \overline{\Lambda^n}$). Conversely, if $\hat{P}^L \subset \overline{\Lambda^n}$, then $\hat{P}^{N(J)} = \varnothing$. Thus, the verification of (8) can be replaced by a verification of the fact that $\mathcal{F}^L(\hat{P}) = \mathcal{F}^L(P)$ or $\mathcal{F}^L(\hat{P}) = \overline{\mathcal{F}}^L(P)$, and to do this Proposition 4.1 can be used (in a simpler situation compared with that at the beginning).

6. A minimal infraction against finiteness of the volume for a non-degenerate convex polytope $P \subset \Lambda^n$ is the existence of a vertex in $\hat{P}$ not in $\overline{\Lambda^n}$ but such that every edge of $\hat{P}$ emanating from it intersects Λ^n. We say that such a vertex of P is *ideal.*

Proposition 4.4. *Let $P \subset \Lambda^n$ be an indecomposable acute-angled polytope and $q = \pi(e)$ an ideal vertex of it. Then the hyperplane H_e of Λ^n intersects orthogonally all edges of P whose continuations pass through q, and $P' = P \cap H_e^-$ is also an indecomposable acute-angled polytope.*

We refer to the transition from P to P' as the *excision of the ideal vertex q.*

Proof. With the system of vectors $\{e_i:\ i \in I\}$ defining P we associate the vector e, normed so that $(e, e) = 1$. The condition $\pi(e) \in \hat{P}$ means that $(e, e_i) \leqslant 0$. Since the e_i, $i \in I$, span $E^{n,1}$, the vector e cannot be orthogonal to all of them, consequently, the system of vectors $\{e_i:\ i \in I\} \cup \{e\}$ is indecomposable. We denote its Gram matrix by G'. By Theorem 2.1, G' is the Gram matrix of some acute-angled polytope. This means that the intersection $P' = P \cap H_e^-$ contains a non-empty open set, hence is a convex polytope, and that the hyperplanes H_i, $i \in I$, and H_e bound this polytope.

Let l be some edge of P whose continuation passes through q. We set $J = \iota\ (l)$. Then G_J is a positive definite matrix of degree $n-1$. Since $(e, e_j) = 0$ for all $j \in J$, the principal submatrix of G' obtained by adjoining rows and columns corresponding to e is also positive definite. By Theorem 3.1, there is a vertex of P' corresponding to it. This vertex is the intersection of l (and of the edge of $\hat{P}$ that is the continuation of it) with H_e. The fact that the continuation of l passes through $q = \pi(e)$ means that it is orthogonal to H_e.

7. We give a sufficient condition for the continuations of n given $(n-1)$-dimensional faces of a polytope P to intersect in an ideal vertex.

A set of $(n-1)$-dimensional faces of P is said to be *connected* if it cannot be split into two non-empty subsets such that any two faces lying in different subsets are not adjacent. (For a simple polytope P it is easy to see that this is equivalent to the topological connectedness of the union of the faces in the given set.)

***Proposition* 4.5.** *Let $P = \bigcap_{i \in I} H_i^- \subset \Lambda^n$ be a non-degenerate acute-angled polytope and $G = G(P)$ its Gram matrix. Let $J \subset I$ be a subset such that*

1) $|J| = n$;

2) *G_J is an indefinite critical matrix;*

3) *the set of $(n-1)$-dimensional faces of P whose indices do not lie in J is connected.*

Then the continuations of the $(n-1)$-dimensional faces whose indices lie in J intersect at an ideal vertex q of P, while the continuations of the other faces do not pass through this vertex.

(This last property means that q is a simplicial vertex of $\hat{P}$).

For the proof we need a property of indefinite critical matrices.

***Lemma* 4.2.** *Let A be an indefinite critical matrix. Then all entries of A^{-1} are negative.*

Proof. Suppose that the order of A is $m+1$. Then A can be regarded as the Gram matrix of some basis $(e_1, \ldots, e_{m+1})$ of the pseudo-Euclidean space $E^{m,1}$, and A^{-1} is the Gram matrix of the dual basis $(f_1, \ldots, f_{m+1})$ of this space. Since every proper principal submatrix of A is positive definite, for any i the subset E_i spanned by all the vectors $(e_1, \ldots, e_{m+1})$ other than e_i is elliptic. Since f_i is orthogonal to this subspace, $(f_i, f_i) < 0$.

Next, the vectors $-f_i$ $(i = 1, \ldots, m+1)$ lie in the cone K defined by the inequalities $(x, e_i) \leqslant 0$ $(i = 1, \ldots, m+1)$. Proposition 2.1 shows that they lie in a single connected component of the cone $C = \{x \in E^{m,1}: (x, x) < 0\}$. Consequently, $(f_i, f_j) < 0$ for all i and j, as required.

Proof of Proposition 4.5. Since G_J is indefinite and non-degenerate, E_J is an n-dimensional hyperbolic subspace of $E^{n,1}$. For $i \in I \setminus J$ we denote by u_i the orthogonal projection of e_i to a (one-dimensional) subspace of E^J. The cone K^J is determined in E^J by the inequalities $(x, u_i) \leqslant 0$, $i \in I \setminus J$. Therefore, the proof of the assertion reduces to proving that all the vectors u_i, $i \in I \setminus J$, are non-zero and directed to the same side.

We have

$$u_i = e_i - \sum_{j,\, k \in J} g_{ij} h_{jk} e_k, \tag{10}$$

where $(h_{jk}) = G_J^{-1}$. By the preceding lemma, $h_{jk} < 0$ for all j and k. Consequently,

$$(u_i, u_l) = (u_i, e_l) = g_{il} - \sum_{j,\, k \in J} g_{ij} h_{jk} g_{kl} \geqslant g_{il} \tag{11}$$

for all $i, l \in I \setminus J$. In particular, $(u_i, u_i) \geqslant 1$, so that $u_i \neq 0$. Hence,

$$(u_i, u_l)^2 = (u_i, u_i)(u_l, u_l) \geqslant 1,$$

that is, $(u_i, u_l) \geqslant 1$ or $(u_i, u_l) \leqslant -1$. In view of (11), the latter case is possible only if $g_{il} \leqslant -1$, that is, if the $(n-1)$-dimensional faces with indices i and l are not adjacent. Thus, if there were vectors among the u_i, $i \in I \backslash J$, in different directions, then the set of $(n-1)$-dimensional faces of P with indices not belonging to J would not be connected, contrary to assumption.

8. By a *simplicial prism* in Λ^n we mean a convex polytope of finite volume whose closure in $\overline{\Lambda^n}$ has the combinatorial type of a simplicial prism. We call a simplicial prism *straight* if one of its bases is orthogonal to every lateral face.

Next, we define a *simplex with k ideal vertices* as a non-degenerate $(n+1)$-tope $P \subset \Lambda^n$ such that all edges of its continuation $\hat{P}$ (which is a simplex) intersect Λ^n, and exactly k of the vertices of $\hat{P}$ do not lie in $\overline{\Lambda^n}$ (and are thus ideal vertices of P).

Let P be an acute-angled simplex with a single ideal vertex q. If the $(n-1)$-dimensional face opposite q is not orthogonal to all the remaining faces, then P is indecomposable, and by Proposition 4.4 the vertex q can be excised. Clearly, this yields a straight acute-angled simplicial prism.

***Proposition* 4.6.** *Every straight acute-angled simplicial prism in Λ^n can be obtained by excising the ideal vertex from a simplex with just one ideal vertex. Every non-straight acute-angled simplicial prism can be obtained by pasting together two straight simplicial prisms along congruent bases orthogonal to the lateral faces.*

Proof. Let $P = \bigcap_{i=1}^{n+2} H_i^-$ be an acute-angled simplicial prism with bases $F_1 = P \cap H_{n+1}$ and $F_2 = P \cap H_{n+2}$. Then $P_1 = \bigcap_{i \neq n+2} H_i^-$ and $P_2 = \bigcap_{i \neq n+1} H_i^-$ are simplexes with a single ideal vertex.

If F_2 is orthogonal to all lateral faces, then P can be obtained by excising an ideal vertex from P_1. If neither of the bases is orthogonal to the lateral faces, then P is the union of two straight prisms obtained by excising ideal vertices from P_1 and P_2 and having common bases orthogonal to the lateral faces.

CHAPTER II

CRYSTALLOGRAPHIC REFLECTION GROUPS IN LOBACHEVSKII SPACES

§5. The language of Coxeter schemes. Construction of crystallographic reflection groups

1. The theory of acute-angled polytopes presented in the first chapter assumes a particularly convenient form when couched in the *language of Coxeter schemes.*

A one-dimensional simplicial complex is called a *graph*, and a *subgraph* of any graph is a subcomplex that contains together with any two adjacent vertices the edge joining them. Further, we define a *scheme* as a graph in which to every edge a positive weight is attached. (Sometimes we speak of an edge of zero weight, meaning by this the absence of the edge.) A *subscheme* of a scheme is a subgraph in which every edge carries the same weight as in the whole scheme. The number of edges of a scheme S is called its order and is denoted by $|S|$.

Let S be a scheme of order n with vertices $v_1, \ldots, v_n$. We form the symmetric matrix (a_{ij}) of degree n in which the diagonal entries are all 1, and a_{ij} for $i \neq j$ is the weight of the edge v_iv_j taken with the minus sign if v_i and v_j are adjacent, and 0 otherwise. We denote this matrix, which is defined up to an equal permutation of the rows and columns, by $A(S)$. It is indecomposable if and only if S is connected.

A scheme S is said to be *elliptic* if $A(S)$ is positive definite, *parabolic* if $A(S)$ is parabolic (see §2.2) and *hyperbolic* if $A(S)$ has index of inertia -1. The *rank* (or *determinant*) of S is that of $A(S)$.

A Coxeter scheme is one in which the weight of every edge is either at least 1 or is of the form $\cos \pi/m$, where m is an integer at least 3. (We may assume that the weight 0 corresponds to the value $m = 1$, and the weight 1 to $m = \infty$.)

Graphically, an edge of a Coxeter scheme can be depicted as follows:

if the weight is $\cos \pi/m$: an $(m-2)$-fold line or a single line marked m;

if the weight is 1: a heavy line or a single line marked ∞;

if the weight is greater than 1: a dotted line marked with the weight (often the mark is omitted).

The number $m-2$ is called the *multiplicity of an edge* of weight $\cos \pi/m$. The multiplicity of an edge of weight at least 1 is taken to be infinite.

2. The *scheme S of an acute-angled polytope P* is defined so that the Gram matrix of P is $A(S)$. In other words, the vertices of S correspond to the hyperplanes bounding P; two vertices are joined by an edge if the corresponding hyperplanes are not orthogonal; the weight an edge is the negative of the cosine of the angle between the hyperplanes if they intersect, -1 if they are parallel, and the negative of the cosine of the distance between them if they are ultraparallel (in the case of Lobachevskii space).

Clearly, the scheme of a polytope P is a Coxeter scheme if and only if P is a Coxeter polytope.

The specification of the scheme of a polytope is equivalent to that of its Gram matrix; however, there are many graphical advantages when the Gram matrix contains several zeros.

By §2.3, the schemes of spherical Coxeter polytopes and of polyhedral Coxeter cones in Euclidean spaces are precisely the Coxeter schemes. The connected elliptic Coxeter schemes are listed in Table 1, together with the accepted designations.

Table 1

Connected elliptic Coxeter schemes (the suffix is the order)

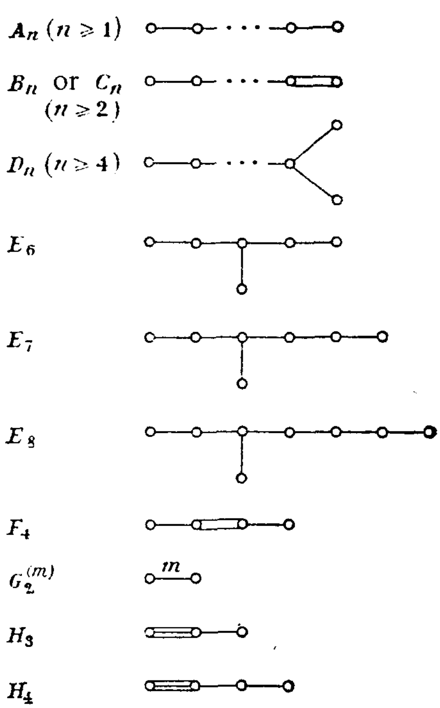
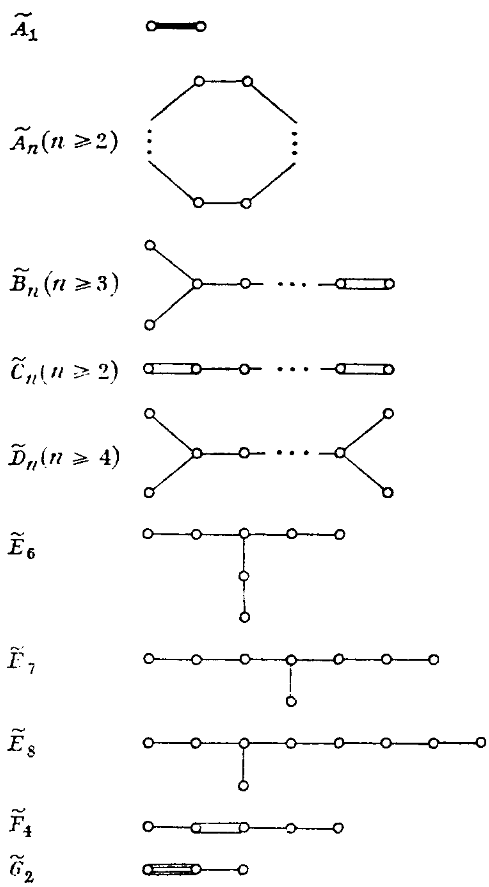


Table 2

Connected parabolic Coxeter schemes (the suffix is the rank)



The schemes of bounded Coxeter polytopes in Euclidean space are exactly the parabolic Coxeter schemes. The connected parabolic Coxeter schemes are listed in Table 2.

3. For Coxeter polytopes in Lobachevskii space the language of schemes makes it possible to reduce the application of Theorems 3.1, 3.2, and 4.1, and Propositions 4.2 and 4.5 to a simple visual procedure.

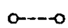
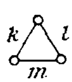
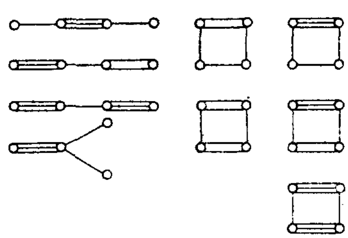
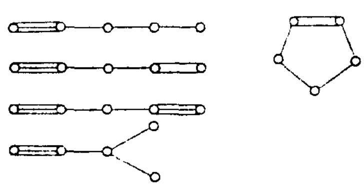
Let $P = \bigcap_{i \in I} H_i^-$ be an acute-angled polytope in n-dimensional Lobachevskii space and S its scheme, so that $A(S) = G$ is the Gram matrix of P. The vertices of S are indexed in the natural way by the elements of I. For any subset $J \subset I$ we denote by S_J the subscheme of S formed from the vertices whose indices lie in J. Clearly, $A(S_J) = G_J$. Consequently, G_J is positive definite (or parabolic) if and only if S_J is elliptic (or parabolic).

Therefore, when P is a Coxeter polytope, the description of the complex $\mathscr{F}(P)$ and the extended complex $\overline{\mathscr{F}}(P)$ reduce by means of Theorems 3.1 and 3.2 to selecting in S subchemes from the known list, namely, the subschemes whose connected components are all in Table 1, and the subschemes of rank $n-1$ whose connected components are all in Table 2. (We note that the rank of such a subscheme is equal to the difference between its order and the number of connected components.)

It follows from Theorem 2.1 and 3.1 that the Gram matrices of bounded simplexes in Λ^n are precisely the indefinite critical matrices of order $n+1$ (see §4.5). These schemes are characterized by the fact that they are neither elliptic nor parabolic, but all of their proper subschemes are elliptic. The Coxeter schemes with these properties are easy to list. We shall call them *Lannér* schemes after F. Lannér, who in 1950 first found all bounded Coxeter simplexes in Lobachevskii spaces [33]. A list of the Lannér schemes is given in Table 3. It is important to note that the orders of Lannér schemes do not exceed 5, that is, bounded Coxeter simplexes in Λ^n exist only for $n \leqslant 4$.

Table 3

Lannér Schemes

orders	schemes
2	
3	 $\left(2 \leqslant k, l, m < \infty,\ \frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1\right)$
4	
5	

We call two subschemes S_1 and S_2 of a scheme S *orthogonal* if no vertex in S_1 is adjacent to any vertex in S_2. For any subset $J \subset I$ the subscheme $S_{I \setminus J}$ is maximal orthogonal to S_J.

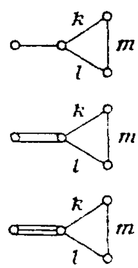
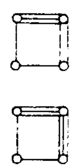
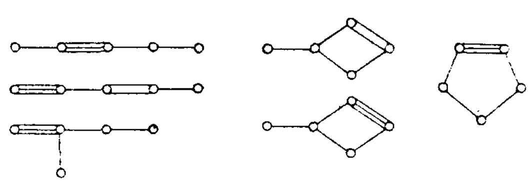
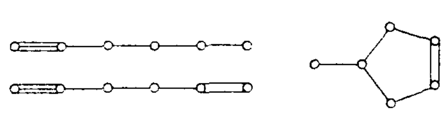
Knowledge of the Lannér schemes and the preceding remarks makes it possible to visualize the applications of Theorem 4.1 to Coxeter polytopes.

4. All simplicial Coxeter prisms in the spaces Λ^n for $n \geqslant 3$ are listed in Kaplinskii's paper [25]. He uses arguments that are essentially the same as in our Proposition 4.6.

The problem reduces to an enumeration of the *straight* simplicial Coxeter prisms, which in turn is equivalent to enumerating the Coxeter simplexes with a single ideal vertex. Such simplexes exist only for $n \leqslant 5$. Their schemes are characterized by the fact that discarding this vertex leaves a Lannér scheme, while discarding any other vertices leaves an elliptic or a connected parabolic scheme. In the classification of bounded prisms the last case must be excluded. The schemes of the corresponding simplexes are given in Table 4. (We note that the list in [25] is incomplete for $n = 3$.)

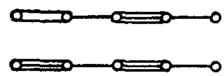
Table 4

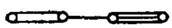
Schemes of Coxeter simplexes with a single ideal vertex, excision of which yields a bounded prism in Λ^n, $n \geqslant 3$

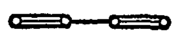
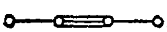
n	Schemes
3	 $(2 \leqslant k, l \leqslant 5,\ \frac{1}{k}+\frac{1}{l}+\frac{1}{m} < 1)$; $(2 \leqslant k, l \leqslant 3\ \ \frac{1}{k}+\frac{1}{l}+\frac{1}{m} < 1)$; $(2 \leqslant k, l \leqslant 3\ \ \frac{1}{k}+\frac{1}{l}+\frac{1}{m} < 1)$ 
4	
5	

In [40], Makarov constructed an infinite sequence of bounded Coxeter polytopes in Λ^4 and Λ^5. In the context of the present paper, these polytopes can be conveniently described as follows.

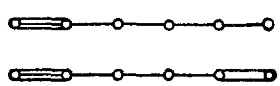
The Coxeter schemes

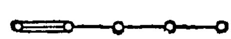


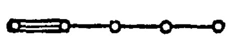
give simplexes in Λ^4 with two ideal vertices. Excision of these vertices yields bounded Coxeter polytopes P_1 and P_2 having each two 3-dimensional simplicial faces orthogonal to all adjacent faces. The types of such faces for P_1 are  and , and for P_2  and .

On forming garlands out of any collection of copies of P_1 and P_2 applied to simplicial faces of the same type, we obtain new Coxeter polytopes. The garland depends not only on the chosen sequence of polytopes, but also on the method of their applications to faces of the types  and , because these faces have symmetries that do not extend to symmetries of the corresponding four-dimensional polytope.

Similarly, the Coxeter schemes



(contained in Table 4) give simplexes in Λ^5 with a single ideal vertex. Excision of these vertices yields bounded simplicial prisms P_1 and P_2. The bases orthogonal to the lateral faces have the type .

The second base of the prism P_2 forms an angle of $\pi/4$ with one of the lateral faces and is orthogonal to the remaining ones. By applying to this face a mirror image of P_2, we obtain a Coxeter polytope P having two simplicial faces of type  orthogonal to all adjacent faces. By forming garlands of any collection of copies of P and by adding copies of P_1 or P_2 to its ends, or adding nothing, we obtain an infinite sequence of Coxeter polytopes in Λ^5.

5. It can be shown that in all examples of bounded Coxeter polytopes in Λ^n, $n \geqslant 3$, given above the number of degrees of freedom of a polytope of given combinatorial type is equal to the number of dihedral angles. We give next a more surprising example, where *the number of degrees of freedom is less than the number of dihedral angles.*

For any scheme S and any vertex v of it we denote by $S-v$ the scheme obtained from S by removing v and the edges emanating from it.

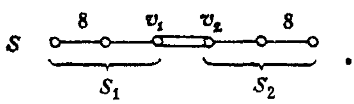
A direct calculation verifies the following result.

***Lemma* 5.1.** *Suppose that an edge v_1v_2 of a scheme S splits it into two parts S_1 and S_2 containing v_1 and v_2, respectively. Then*

$$\det S = \det S_1 \cdot \det S_2 - a^2 \det(S_1 - v_1)\cdot\det(S_2 - v_2),$$

where a is the weight of v_1v_2.

We consider now the following Coxeter scheme of order 6:

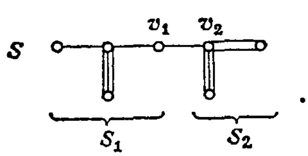


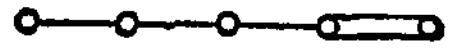
The edge v_1v_2 splits it into two Lannér subschemes S_1 and S_2. By means of Lemma 5.1 we find that $\det S = 0$. The scheme S is not parabolic, but it contains elliptic subschemes of order 4, for example, .

Consequently, S is a hyperbolic scheme of rank 5. By Theorem 2.1, it is the scheme of a Coxeter polytope $P \subset \Lambda^4$.

Every subscheme of S containing neither of the Lannér subschemes S_1 and S_2 is elliptic. In view of Theorem 3.1, this means that the complex $\mathscr{F}(P)$ is isomorphic to the complex of a direct product of two triangles (see §3.1). Applying Proposition 4.2, we obtain from this that P is a bounded polytope combinatorially isomorphic to the direct product of two triangles. The number of degrees of freedom of this polytope is $4.6-10 = 14$ (see the footnote (1) on p.35), whereas the number of dihedral angles is $\binom{6}{2} = 15$.

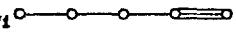
Next we consider the following Coxeter scheme of order 7:

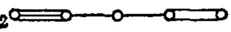


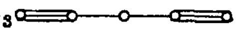
The edge v_1v_2 splits it into two Lannér subschemes S_1 and S_2. By means of Lemma 5.1 we find that $\det S = 0$. The scheme S is not parabolic, but contains elliptic subschemes of order 5, for example .

Consequently S is a hyperbolic scheme of rank 6. It is the scheme of a Coxeter polytope $P \subset \Lambda^5$.

Every subscheme of S containing neither of the Lannér subschemes S_1 and S_2 and different from the Lannér subschemes

L_1 

L_2 

L_3 

is elliptic. The subschemes L_1, L_2, and L_3 satisfy the conditions of Proposition 4.5, so that there are ideal vertices q_1, q_2, and q_3 of P corresponding to them. The segment of $\mathscr{F}(\hat{P})$ obtained by adding to $\mathscr{F}(P)$ the subsets corresponding to these three ideal vertices is isomorphic to the complex of the direct product of a tetrahedron and a triangle, hence, is the whole of $\mathscr{F}(\hat{P})$ (Proposition 4.1). Thus, $\hat{P}$ is combinatorially isomorphic to the direct product of a tetrahedron and a triangle, and all its faces except the vertices q_1, q_2, and q_3 have non-empty intersection with Λ^5. Excising these ideal vertices in accordance with Proposition 4.4, we obtain a bounded Coxeter polytope $P' \subset \Lambda^5$ having 10 four-dimensional faces. The number of degrees of freedom of this polytope is $5.10-15 = 35$, while the number of dihedral angles is $\binom{7}{2} + 3\cdot 5 = 36$.

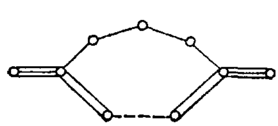
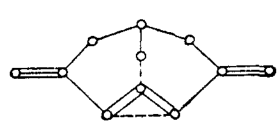
The polytope P' has three simplicial four-dimensional faces of type L_1, L_2, and L_3 orthogonal to all adjacent faces. Applying various copies of P' to one another according to simplicial faces of the same type, we can obtain infinitely many tree-like Coxeter polytopes. Among the pieces of this construction set we can include the two Makarov prisms (see §5.4), each of which has a four-dimensional face of type L_1 orthogonal to all adjacent faces.

6. The highest dimension of a Lobachevskii space for which bounded Coxeter polytopes are known to exist is 7. Examples in Λ^6 and Λ^7 were found by Bugaenko as fundamental polytopes P_6 and P_7 of the reflection subgroups in groups of integral linear transformations over the field $\mathbb{Q}(\sqrt{5})$ preserving the quadratic form

$$-\frac{\sqrt{5}+1}{2}x_0^2 + x_1^2 + \dots + x_n^2$$

for $n = 6$ and $n = 7$, respectively.

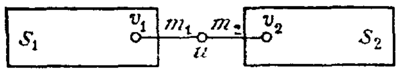
The schemes of these polytopes are:

  .

That P_6 and P_7 are bounded is easy to prove by means of Theorem 4.1. For both the schemes depicted have no parabolic subschemes, and (9) holds for any of their Lannér subschemes.

7. We now describe a *general construction of unbounded Coxeter polytopes of finite volume* in Lobachevskii spaces.

Let S_1 and S_2 be connected parabolic Coxeter schemes of rank n_1 and n_2, respectively. We consider the Coxeter scheme



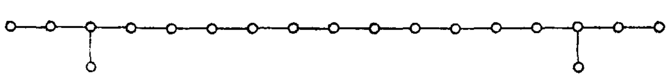
$(m_1, m_2 \geq 3)$,

in which u is joined to exactly one vertex of each of the subschemes S_1 and S_2. The edge uv_2 splits it into two subschemes: the subscheme T obtained by adjoining to S_1 the vertex u, and S_2 itself. Since $T-u = S_1$ and $\det S_1 = \det S_2 = 0$, we deduce from that from Lemma 5.1, applied to the edge uv_2 that $\det S = 0$. At the same time, S is not parabolic, but contains the elliptic subscheme of order $n = n_1+n_2+1$ obtained by excising v_1 and v_2. Consequently, S is a hyperbolic scheme of rank $n+1$ (and of order $n+2$) and is the scheme of a Coxeter polytope $P \subset \Lambda^n$.

The subscheme $S-u$ is parabolic of rank $n_1+n_2 = n-1$. By Theorem 3.2, there corresponds to it a vertex q at infinity of P. The intersection of P with a small horosphere with centre at q is a convex polytope in $(n-1)$-dimensional Euclidean space whose scheme is $S-u$ and which is therefore the direct product of simplexes of dimensions n_1 and n_2 with schemes S_1 and S_2, respectively.

The polytope $\hat{P}$ is a pyramid with vertex at q whose base lies in the hyperplane of the projective sphere corresponding to the vertex u of S and combinatorially constructed as the direct product of simplexes of dimensions n_1 and n_2. The vertices of $\hat{P}$ lying in the bases of the pyramid correspond to the subschemes of S obtained by removing one vertex each from S_1 and S_2. If all the resulting subschemes are elliptic or parabolic, then $\hat{P} \subset \overline{\Lambda}^n$, hence, P has finite volume.

There are many schemes of the relevant form that satisfy this last condition. Among them the scheme of highest order is



(in this case S_1 and S_2 are of type $\widetilde{E}_8$). It is the scheme of an unbounded Coxeter polytope in 17-dimensional Lobachevskii space. Its closure can be constructed combinatorially as a pyramid over a direct product of two 9-dimensional simplexes.

8. It is proved in [14] and [20] (see also [44]) that the reflection subgroup of the group $O_{n,1}(\mathbf{Z})$ of integral linear transformations preserving the quadratic form

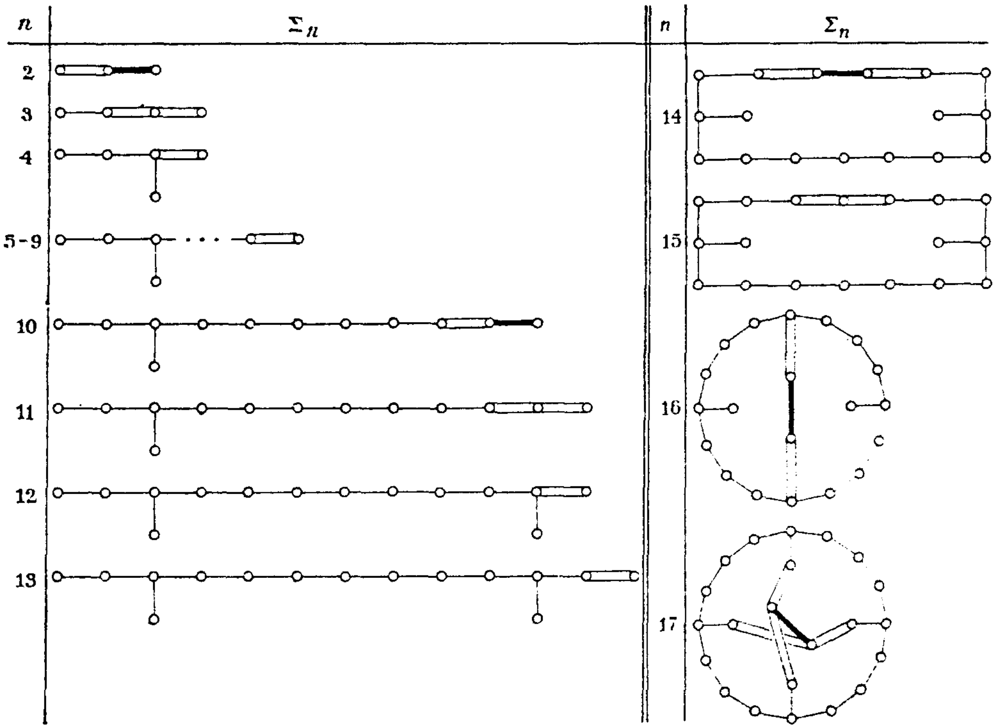
$$-x_0^2+x_1^2+\ldots+x_n^2,$$

has finite index for $n \leqslant 19$, and the fundamental polytopes P_n $(2 \leqslant n \leqslant 19)$ of these subgroups are determined there.

The schemes Σ_n of the P_n for $2 \leqslant n \leqslant 17$ are given in Table 5.

Table 5

Schemes of fundamental polytopes of reflection subgroups of the groups $O_{n,1}(\mathbf{Z})$



When we look at these schemes, we see that the P_n for $n \leqslant 9$ are simplexes. For $10 \leqslant n \leqslant 13$ the Σ_n belong to the type described in 7, and the P_n are pyramids over direct products of two simplexes. The combinatorial structure of the P_n for $14 \leqslant n \leqslant 17$ does not lend itself to a simple verbal description. That their volume is finite is established by means of Theorem 4.1. Inspection shows that their schemes do not contain Lannér subschemes and that every connected parabolic subscheme is contained (as a connected component) in a parabolic subscheme of rank $n-1$.

The polytopes P_{18} and P_{19} have a considerable more complicated structure: the first has 37 and the second 50 faces of codimension 1. Their schemes are described in [20]. That the volume of these polytopes is finite is established by means of Theorem 4.1.

§6. The non-existence of discrete reflection groups with bounded fundamental polytopes in higher-dimensional Lobachevskii spaces

1. The classification problem for discrete reflection groups in Lobachevskii spaces differs from those in Euclidean space and on the sphere principally in that even bounded Coxeter polytopes in Lobachevskii space may have a complicated combinatorial structure. To stydy them we need some information on the combinatorial properties of convex polytopes.

At present the only known general combinatorial property of Coxeter polytopes is that they are *simple* (see §3.1). Some results on the number of faces of bounded simple polytopes are stated below(1). They play a key role in the proof of the main theorem in this section.

Let P be a bounded n-dimensional simple convex polytope. We denote by a_k $(0 \leqslant k \leqslant n)$ the number of its k-dimensional faces. Then

$$\sum_{k=0}^{n} a_k (t-1)^k = \sum_{k=0}^{n} b_k t^k,$$

is called the *combinatorial polynomial* of P. Its coefficients have the following properties:

$$b_k = b_{n-k} \geqslant 0 \quad (k = 0, 1, \ldots, n). \tag{12}$$

(The fact that $b_k = b_{n-k}$ is known as the *Dehn-Sommerville equality*; this has been known for a long time [56]. The inequalities $b_k \geqslant 0$ were proved only recently [57], [22]).

We set $m = [n/2]$. By the Dehn-Sommerville equalities, the numbers a_k can be expressed in terms of the coefficients $b_1, \ldots, b_m$ of the combinatorial polynomial as follows:

$$a_k = \sum_{p=0}^{m} \left(\binom{n-p}{k} + \binom{p}{k} \right) \hat{b}_p,$$

where

$$\hat{b}_p = \begin{cases} b_p & \text{for} \quad p \neq \frac{n}{2}, \\ \frac{1}{2} b_p & \text{for} \quad p = \frac{n}{2}. \end{cases}$$

2. In his paper [49] Nikulin derives from (12) an upper bound for the average number of l-dimensional faces of a k-dimensional face of an n-dimensional simple polytope for $l < k \leqslant (n+1)/2$. We need only the special case $k = 2$, $l = 0$ of this bound.

(1) As a rule, these results are stated for simplicial polytopes, but by duality they carry over automatically to simple polytopes.

***Proposition* 6.1.** *The average number of vertices of a two-dimensional face of a bounded n-dimensional simple convex polytope P is less than*

$$\begin{cases} \dfrac{4(n-1)}{n-2} & \text{for even } n, \\ \dfrac{4n}{n-1} & \text{for odd } n. \end{cases}$$

Proof. We have

$$a_0 = 2\sum_{p=0}^{m} \hat{b}_p, \quad a_2 = \sum_{p=0}^{m} \left(\binom{n-p}{2} + \binom{p}{2}\right)\hat{b}_p.$$

Since for $p \leqslant m$

$$\left(\binom{n-(p-1)}{2} + \binom{p-1}{2}\right) - \left(\binom{n-p}{2} + \binom{p}{2}\right) =$$
$$= \binom{n-p}{1} - \binom{p-1}{1} = n-2p+1 > 0,$$

the coefficients in the expression for a_2 are decreasing, hence,

$$a_2 > \begin{cases} 2\binom{m}{2}\sum_{p=0}^{m} \hat{b}_p & \text{for even } n, \\ \left(\binom{m+1}{2} + \binom{m}{2}\right)\sum_{p=0}^{m} \hat{b}_p & \text{for odd } n. \end{cases}$$

The average number of vertices of a two-dimensional face is a_0/a_2 multiplied by the number of two-dimensional faces passing through each vertex of P, that is, by $\binom{n}{2}$. Consequently, this number is less than

$$\frac{\binom{n}{2}}{\binom{m}{2}} = \frac{4(n-1)}{n-2} \quad \text{for even } n,$$

$$\frac{2\binom{n}{2}}{\binom{m+1}{2} + \binom{m}{2}} = \frac{4n}{n-1} \quad \text{for odd } n.$$

From Proposition 6.1 it follows, in particular, that for $n \geqslant 5$ the polytope P always has at least one quadrilateral or triangular two-dimensional face.

3. A *planar angle* of a polytope P is a pair (A, F), where A is a vertex and F a two-dimensional face containing it. We say that (A, F) is a planar angle at A and also a planar angle of F.

***Proposition* 6.2.** *Let P be a bounded n-dimensional simple convex polytope and c a positive number. We assume that the planar angles of P can be endowed with weights in such a way that*

(a) *the sum $\sigma(A)$ of the weights of the planar angles at the vertex A does not exceed cn*;

(b) *the sum $\sigma(F)$ of the weights of the planar angles of any two-dimensional face is not less than $5-k$, where k is the number of vertices of this face.*

Then $n < 8c+6$.

Proof. Let $\varkappa = \binom{n}{2} a_0/a_2$ be the average number of vertices of a two-dimensional face of P. We estimate the sum σ of the weights of all planar angles in two ways. It follows from a) that

$$\sigma = \sum_A \sigma(A) \leqslant cna_0 = \frac{2c\varkappa}{n-1} a_2,$$

and from b) that

$$\sigma = \sum_F \sigma(F) \geqslant \sum_k (5-k) a_{2,k} = (5-\chi) a_2,$$

where $a_{2,k}$ is the number of k-angled two-dimensional faces. Combining these inequalities we find that

$$5 - \varkappa \leqslant \frac{2c\varkappa}{n-1}. \tag{13}$$

By Proposition 6.1, $\varkappa < 4(n-1)/(n-2)$ for even n, and from (13) we obtain that

$$n - 6 < 8c, \tag{14}$$

as required. For odd n we have $\varkappa < 4n/(n-1)$, and from (13) we obtain $(n-1)(n-5) < 8cn$; now (14) follows, since $(n-1)(n-5) > n(n-6)$.

4. We claim that there are no bounded Coxeter polytopes in Lobachevskii space of dimension at least 62. This leads to the following theorem.

***Theorem* 6.1.** *There are no discrete reflection groups with bounded fundamental polytopes in Lobachevskii space of dimension at least* 62.

The proof of this theorem is based on Proposition 6.2. By the same method but technically in a much more complicated way the author has proved in [18] that even in a Lobachevskii space of dimension at least 30 there are no bounded Coxeter polytopes.

5. Let P be a bounded acute-angled polytope in n-dimensional Lobachevskii space, and S its scheme (see §5.3). Since P is indecomposable (see §1.5), S is a connected hyperbolic scheme. It follows from Proposition 3.2 that it contains no parabolic subschemes. We mention also that it cannot contain two orthogonal hyperbolic subspaces (see §5.4), since then the negative of the index of inertia of $A(S)$ would be more than 1.

For every face F of P we denote by S_F the subscheme of S whose vertices correspond to the $(n-1)$-dimensional faces containing F. This is an elliptic scheme of order $n - \dim F$.

The *scheme of a planar angle* (A, F) of a polytope P is the scheme S_A with two chosen "black" vertices corresponding to the $(n-1)$-dimensional faces not containing F. This is an elliptic scheme of order n.

The *star scheme of a face* F of a polytope P is the subscheme S_F^* of S whose vertices correspond to the $(n-1)$-dimensional faces having common points with F. We call the vertices of S_F^* that belong to S_F "white", and the remaining vertices "black". The black vertices correspond to the $(n-1)$-dimensional faces that intersect F in faces of codimension 1 (in F).

In what follows we consider the star schemes of quadrilateral and triangular two-dimensional faces. We mention some of their properties.

***Proposition* 6.3.** *Let S_F^* be the star scheme of a triangular two-dimensional face F of a polytope P. Then*

1) *removal from S_F of any black vertex yields the scheme of one of the planar angles of F;*

2) *every hyperbolic subscheme of S_F^* contains all three black vertices.*

Proof. 1) is obvious. Since the scheme of a planar angle is elliptic, 2) follows from 1).

6. ***Proposition* 6.4.** *Let S_F^* be the star scheme of a quadrilateral two-dimensional face F of a polytope P. We divide the black vertices of S_F^* into pairs in such a way that the vertices corresponding to opposite sides of F correspond to a single pair. Then*

1) *removal from S_F^* of any two black vertices in different pairs leaves the scheme of one of the planar angles of F;*

2) *every hyperbolic subscheme of S_F^* contains both the black vertices from some pair;*

3) *removal from S_F^* of the two black vertices in the same pair leaves a hyperbolic scheme.*

Proof. 1) is obvious; 2) follows from 1); 3) follows from the fact that the $(n-1)$-dimensional faces of P corresponding to the white vertices of S_F^* and the two black vertices of the remaining pair have no common points.

7. We call a sequence of distinct vertices $v_0, v_1, ..., v_m$ of a graph in which v_{k-1} and v_k are adjacent $(k = 1, ..., m)$, together with the sequence of edges $v_0v_1, v_1v_2, \dots, v_{m-1}v_m$ a *path* of length m joining v_0 and v_m. The length of a shortest path joining two vertices is the *distance* between them. If two vertices cannot be joined by a path, we take the distance between them to be infinite.

***Lemma* 6.1.** *For every elliptic Coxeter scheme T of order n the number $f(T, c)$ of (unordered) pairs of vertices at a distance at most c is at most cn.*

Proof. If T is a disjoint union of schemes T_1 and T_2, then $f(T, c) = f(T_1, c)+f(T_2, c)$. Therefore, we need prove the assertion only for connected schemes. Moreover, it is automatically true if $n \leqslant 2c+1$, since then the number of all pairs of vertices is not more than cn.

Now let T be a connected elliptic scheme of order $n \geqslant c+1$. For any $d \leqslant c$ the number of pairs of vertices at a distance d is $n-d$ if the scheme is linear, and is at most $n-d+1$ if it has a "shoot". In each case the number is at most n, and the assertion follows.

8. *Proof of Theorem* 6.1. Let P be a bounded Coxeter polytope in n-dimensional Lobachevskii space and S its scheme.

We attach weights to the planar angles of P as follows: the weight of a planar angle is 1 if the black vertices in its scheme are at a distance of at most 7, and 0 otherwise. We check that the hypotheses of Proposition 6.2 are satisfied with $c = 7$, which implies that $n < 62$.

Now a) follows from Lemma 6.1; b) needs checking for two-dimensional triangular and quadrilateral faces.

Let F be a two-dimensional triangular face of P. It follows from Proposition 6.3 that all three black vertices of S_F^* are contained in some Lannér subscheme L. Since L is connected, it has at most one black vertex seperating the other two. In other words, there are at least two black vertices v such that for each of them two other black vertices are contained in a single connected component of the scheme $L-v$. Since $|L-v| \leqslant 4$, the distance between black vertices in $L-v$ is at most 3, and hence the same is true of the scheme of the corresponding planar angle. Thus, we have shown that $\sigma(F) \geqslant 2$.

Suppose now that F is a two-dimensional quadrilateral face of P. It follows from Proposition 6.4 that every pair of black vertices of S_F^* is contained in some Lannér subscheme not containing the black vertices of the other pair. Let L and M be the chosen Lannér subschemes. They cannot be orthogonal, that is, they must either have a common vertex or be joined by an edge. Since L and M are connected schemes, we can find black vertices u and v from distinct pairs that can be joined in $L \cup M$ by a path not involving other black vertices. Since the number of white vertices of $L \cup M$ is at most 6, the length of this path is at most 7. Thus, the black vertices u and v lie in the scheme of the corresponding planar angle of F at a distance of at most 7. Thus, we have shown that $\sigma(F) \geqslant 1$.

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Translated by J. Wiegold

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Received by the Editors 6 January 1983