JOURNAL OF ALGEBRA **80**, 159–163 (1983)

The Automorphism Group of the 26-Dimensional Even Unimodular Lorentzian Lattice

J. H. CONWAY

Department of Pure Mathematics and Mathematical Statistics,

University of Cambridge, Cambridge CB2 1SB, England

Communicated by W. Feit

Received March 10, 1982

The group mentioned in the title is shown to be a certain infinitely generated Coxeter group extended by the negative of the identity operation and the group of all automorphisms of the notorious Leech lattice.

Let $\mathbb{R}^{n,1}$ denote the $(n+1)$-dimensional real vector space of vectors $x = (x_1, x_2, ..., x_n | x_0)$ equipped with the Lorentzian norm

$$N(x) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_0^2.$$

For all $n \geqslant 1$ there is a unique odd (or Type I) unimodular lattice $\mathrm{I}_{n,1}$ in $\mathbb{R}^{n,1}$, which can be taken to be the set of points x for which all x_i are in $\mathbb{Z}$ (the integers). In a series of papers Vinberg, and Vinberg and Kaplinskaja [3–6] have shown that the reflection subgroup of $\mathrm{Aut}(\mathrm{I}_{n,1})$ has finite index only if $n \leqslant 19$, and have given Coxeter graphs for the reflection subgroups in these cases.

When $n \equiv 1$ (modulo 8), there is a second unimodular lattice in $\mathbb{R}^{n,1}$, namely, the even (or Type II) lattice $\mathrm{II}_{n,1}$, which can be taken to be the set of x for which the x_i are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector

$$(\tfrac{1}{2}, \tfrac{1}{2}, ..., \tfrac{1}{2} | \tfrac{1}{2}).$$

In [5] Vinberg has also shown that the reflection subgroups of $\mathrm{Aut}(\mathrm{II}_{9,1})$ and $\mathrm{Aut}(\mathrm{II}_{17,1})$ have finite index and are described by the Coxeter graphs shown in Figs. 1 and 2.

As is customary, we define a *root* for a lattice L to be a vector r of positive norm for which the associated reflection

$$x \to x - 2\frac{x \cdot r}{r \cdot r} r$$

0021-8693/83/010159–05$03.00/0

481/80/1-11

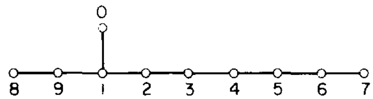

FIG. 1. The Coxeter graph for $\mathrm{II}_{9,1}$. The points are: i: $(0^i, +1, -1, 0^{7-i}|0)$, for $0 \leqslant i \leqslant 7$; 8: $((\frac{1}{2})^9|\frac{1}{2})$; 9: $(-1^2, 0^7|0)$.

is a symmetry of L. If L is integral and unimodular the roots are just the vectors of norms 1 and 2 in L. Now the vectors of positive time coordinate x_0 and negative or zero norm in a Lorentzian space, taken modulo positive scalar factors, become the ordinary and ideal points, respectively, of the associated hyperbolic space. The symmetry group of the hyperbolic space can be identified with the autochronous symmetries of the Lorentzian space (those symmetries not interchanging the positive and negative time cones). The full group of symmetries of the Lorentzian space is just the autochronous subgroup extended by the negative of the identity operation.

If L is a Euclidean or Lorentzian lattice, the reflecting hyperplanes corresponding to all the roots of L partition the corresponding Euclidean or hyperbolic space into fundamental regions. The roots corresponding to the walls of any one fundamental region are a set of *fundamental roots*, for which the corresponding reflections generate the smallest group containing all reflections in L. The latter group we call the reflection group of L.

In this paper we prove the following theorem.

THEOREM. *For* $n = 9, 17, 25$ *there are respectively* 10, 19, ∞ *fundamental roots for* $\mathrm{II}_{n,1}$, *which can be taken to be the vectors r with*

$$r \in \mathrm{II}_{n,1}, \qquad r \cdot r = 2, \qquad r \cdot w_n = -1, \tag{*}$$

where

$$w_9 = (0, 1, \ldots, 8|38), \qquad N(w_9) = -1240,$$
$$w_{17} = (0, 1, \ldots, 16|46), \qquad N(w_{17}) = -620,$$
$$w_{25} = (0, 1, \ldots, 24|70), \qquad N(w_{25}) = 0.$$

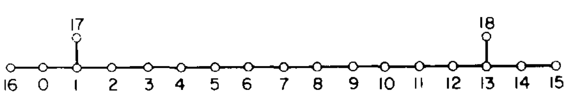

FIG. 2. The Coxeter graph for $\mathrm{II}_{17,1}$. The points are: i: $(0^i, +1, -1, 0^{15-i}|0)$, for $0 \leqslant i \leqslant 15$; 16: $(-\frac{1}{2}, (\frac{1}{2})^{16}|\frac{3}{2})$; 17: $(-1^2, 0^{15}|0)$; 18: $(0^{14}, 1^3|1)$.

The group of all autochronous automorphisms of $\mathrm{II}_{n,1}$ *is the Coxeter group generated by the corresponding reflections, extended by*

for $n = 9$, *the trivial group,*

for $n = 17$, *a group of order* 2, or

for $n = 25$, *an infinite group abstractly isomorphic to the group* $\cdot\infty$ *of all automorphisms of the Leech lattice (including translations).*

The corresponding Coxeter graphs are displayed for $n = 9$ *in Fig.* 1 *and for* $n = 17$ *in Fig.* 2. *For* $n = 25$, *a portion of the graph is shown in Fig.* 3. *The full graph has one node for each Leech lattice vector and (using Vinberg's conventions) two nodes r, s are joined by*

no line if $N(r - s) = 4$,

an ordinary line if $N(r - s) = 6$,

a heavy line if $N(r - s) = 8$, *or*

a broken line if $N(r - s) \geqslant 10$.

Proof. It is easy to check that for $n = 9$ and 17 all r satisfying the conditions (*) are displayed in Figs. 1 and 2. Since the graphs of these figures agree with those given by Vinberg [5, p. 347], the assertion of the theorem holds for $n = 9$ and 17.

For $n = 25$ it was pointed out in [2] that there is a one-to-one correspondence between the points of the Leech lattice and the vectors r satisfying (*),

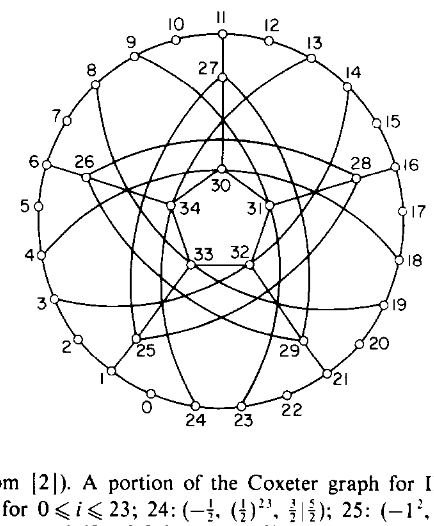

FIG. 3. (Taken from [2]). A portion of the Coxeter graph for $\mathrm{II}_{25,1}$. The points are: i: $(0^i, +1, -1, 0^{23-i}|0)$, for $0 \leqslant i \leqslant 23$; 24: $(-\frac{1}{2}, (\frac{1}{2})^{23}, \frac{3}{2}|\frac{5}{2})$; 25: $(-1^2, 0^{23}|0)$; 26: $(0^7, 1^{18}|4)$; 27: $((\frac{1}{2})^{12}, (\frac{3}{2})^{13}|\frac{11}{2})$; 28: $((\frac{1}{2})^{17}, (\frac{3}{2})^8|\frac{9}{2})$; 29: $(0^{22}, 1^3|1)$; 30: $(0^5, 1^{14}, 2^6|6)$; 31: $(0^{10}, 1^{14}, 2|4)$; 32: $(0^4, 1^{11}, 2^{10}|7)$; 33: $((\frac{1}{2})^9, (\frac{3}{2})^{11}, (\frac{5}{2})^5|\frac{15}{2})$; 34: $(0^{14}, 1^{11}|3)$.

which we propose to call *Leech roots*. Indeed this correspondence is an isometry for the metric defined by

$$d(r, s)^2 = N(r - s).$$

The assertion of the theorem will therefore follow if we can show that the Leech roots are precisely the fundamental roots for $II_{25,1}$ since they have the joins and symmetries indicated in the statement of the theorem.

We show this using the algorithm described by Vinberg for finding the fundamental roots for any discrete hyperbolic reflection group [5]. We take Vinberg's vector x_0 to be w_{25} (which satisfies his conditions), and define the height of a vector r to be $h(r) = -r \cdot w_{25}$. The algorithm then proceeds as follows. The vectors in $II_{25,1}$ of norm 1 or 2 (for us, only norm 2) and positive height are enumerated in increasing order of height · norm$^{-1/2}$ (for us, in order of height). Vectors of height zero have norm at least 4, by the result of [2], and so need not be considered in our case. (In more general cases such vectors require special treatment when starting the algorithm.) A vector is rejected by the algorithm if it has strictly positive inner product with any previously accepted vector; otherwise it is accepted as a fundamental root. It is not necessary to test the condition between vectors of the same height.

It is clear that in our case all vectors of height 1 and norm 2 are accepted. We shall show that every other norm 2 candidate x is rejected.

Let $h(x) = h$, and define $v = x/h$. Then v lies in the affine hyperplane of height 1 vectors and so by [2] specifies a point in the rational 24-dimensional space of the Leech lattice. By [1] there is a Leech lattice vector r, which we can regard as a Leech root, such that $N(v - r) \leqslant 2$. Thus

$$\frac{2}{h^2} - \frac{2}{h} x \cdot r + 2 \leqslant 2$$

or

$$x \cdot r \geqslant \frac{1}{h},$$

showing that x is rejected.

Remarks. (1) Several alternative proofs are known, all of which, however, depend on the main theorem of [1]. Since that theorem has only been proved by laborious calculations, it would be desirable to find a simpler proof.

(2) It can be shown that for $n = 33, 41, \ldots$ there is no vector w_n having constant inner product with the fundamental roots for $II_{n,1}$.

(3) The group $\cdot\infty$ of graph automorphisms acts transitively on the fundamental roots! Compare $II_{9,1}$, where the graph has no (nontrivial) automorphism, and $II_{17,1}$, where the automorphism group has order 2. I do not know whether the group is transitive for any of $II_{33,1}$,....

(4) Is there any connection with the Fischer–Griess Monster group?

ACKNOWLEDGMENTS

Part of this research was carried out at Bell Laboratories in January 1982, and the author wishes to thank Bell Labs for support and hospitality.

REFERENCES

1. J. H. CONWAY, R. A. PARKER, AND N. J. A. SLOANE, The covering radius of the Leech lattice, *Proc. Roy. Soc. London Ser. A* **380** (1982), 261–290.
2. J. H. CONWAY AND N. J. A. SLOANE, Lorentzian forms for the Leech lattice, *Bull. Amer. Math. Soc.* **6** (1982), 215–217.
3. È. B. VINBERG, Discrete groups generated by reflections in Lovačevskii spaces, *Mat. Sb.* **72**, No. 3 (1967), 471–488. [English translation: *Math. USSR Sb.* **1**, No. 3 (1967), 429–444.]
4. È. B. VINBERG, On the groups of units of certain quadratic forms, *Mat. Sb.* **87**, No. 1 (1972), 18–36. [English translation: *Math. USSR Sb.* **16**, No. 1 (1972), 17–35.]
5. È. B. VINBERG, Some arithmetical discrete groups in Lobačevskii spaces, *in* "Discrete Subgroups of Lie Groups and Applications to Moduli," pp. 323–348, Oxford Univ. Press, Oxford, 1975.
6. È. B. VINBERG AND I. M. KAPLINSKAJA, On the groups $O_{18,1}(\mathbb{Z})$ and $O_{19,1}(\mathbb{Z})$, *Dokl. Akad. Nauk SSR* **238**, No. 6 (1978). [English translation: *Soviet Math. Dokl.* **19**, No. 1 (1978), 194–197.