DEFINABILITY IN THE FOUNDATIONS OF EUCLIDEAN GEOMETRY AND THE PRODUCT RULE FOR DERIVATIONS

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Abstract. In this talk, we discuss the results of investigations that began with a solution to an open problem posed by Schwabhäuser and Szczerba regarding definability (without parameters) in the three dimensional Euclidean geometry of lines, asking whether intersection was definable from perpendicularity (two lines intersecting at a right angle). It is not. The result is a "new" 3-dimensional geometry of lines, which we call *perpendicular geometry*, since it can be formalized from perpendicularity. Further investigations produce a rather complete classification of possible geometries arising from elementary Euclidean binary relations between lines in $\mathbb{R}^3$, modulo the determination of (metrical) projective geometries formalized by binary relations between points. The classification shows that in a sense made precise, perpendicular geometry is the only new geometry that can arise from binary geometric relations, except for possible new projective plane geometries, which we conjecture do not exist. Generalizing to geometries of s-flats in n-dimensional Euclidean geometry, we state a theorem which provides the essential first step towards a similar classification for the general case. The theorem states that parallel is definable in such a geometry no matter what binary geometric relation that one might chose to formalize geometry (with an enumerated list of trivial exceptions). To what extent this is true for ternary or higher order geometric relations is open, even for lines in $\mathbb{R}^3$. We conjecture that it remains true, that is, that parallel is definable from *anything* "except for the exceptions". Finally, we note that perpendicular geometry, whose automorphism group is connected with derivations, sheds some rather curious light on the relationship between the product rule for derivations over a ring, and the sum rule. For example, it is a direct consequence of perpendicular geometry that the product rule for the cross product of 3-dimensional vectors implies the sum rule. It is conjectured that this is also true of all finite dimensional semi-simple Lie algebras over the complex numbers.

In [2], Schwabhäuser and Szczerba investigated the possibility of formulating Euclidean geometry based on lines as primitive notions, together with geometric relations between lines. (An n-ary relation between lines in $\mathbb{R}^3$ is called *geometric* provided that the relation is preserved under similarity transformations, that is, compositions of rotations, reflections, dilations, and translations.) They showed that for any $n \geq 2$, the binary relation of perpendicularity (two lines intersecting at a right angle), together with the ternary relation of copunctuality (three lines intersecting at a single point) suffices to formalize n-dimensional Euclidean geometry. (Essentially, these relations suffice to interpret the points as equivalence classes of pairs of intersecting lines.) They went on to show that for $n \geq 4$, the single binary relation of perpendicularity sufficed, since copunctuality is definable from perpendicularity for $n \geq 4$. (Throughout this abstract, by "definable" we mean first order definable *without parameters*, unless otherwise stated.)

It is easy to describe the binary geometric relations on $\mathbf{L}_1(\mathbb{R}^n)$. They are the unions of orbits of the action of the group of similarity transformations on $\mathbf{L}_1(\mathbb{R}^n)$. Let $\times^\bullet_\theta$ be the binary relation of two lines intersecting at angle θ, and let $\times^\circ_\theta$ be the binary relation of two lines being skew at angle θ (parallel but not equal if $\theta = 0$). If $\mathcal{A} \subseteq \mathbb{R}$, we define $\times^\bullet_\mathcal{A} = \bigcup_{\theta\in\mathcal{A}} \times^\bullet_\theta$, and $\times^\circ_\mathcal{A} = \bigcup_{\theta\in\mathcal{A}} \times^\circ_\theta$. A general binary geometric relation $\mathbf{R}$ on $\mathbf{L}_1(\mathbb{R}^n)$ has the form $\mathbf{R} = \times^\bullet_\mathcal{A} \cup \times^\circ_\mathcal{B}$ where $\mathcal{A}, \mathcal{B} \subseteq [0, \frac{\pi}{2}]$ for $n \geq 3$. For $n = 2$, we further require that $\mathcal{B} \subseteq \{0\}$, since skew lines do not exist in $\mathbb{R}^2$.

Schwabhäuser and Szczerba went on to observe that for $n = 2$, the binary relation of perpendicularity and the ternary relation of copunctuality formed a minimal set of relations between lines to formalize Euclidean geometry. In fact, they observed that *no* set of binary relations would suffice in that case, so the use of a ternary (or higher) relation was essential. (The proof is trivial. Simply choose two distinct parallel lines, and define the bijection $\sigma : \mathbf{L}_1(\mathbb{R}^2) \to \mathbf{L}_1(\mathbb{R}^2)$ that transposes the two lines, and leaves everything else fixed. This bijection preserves all the the binary geometric relations, but does not preserve the ternary relation of copunctuality.)

This left the case of $n = 3$. In this case, they noted that the binary relation of perpendicularity and the binary relation of intersection sufficed, since the ternary relation of copunctuality is definable in $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet, \times^\bullet \rangle$, where $\perp^\bullet = \times_{\frac{\pi}{2}}$ and $\times^\bullet = \times^\bullet_{[0,\frac{\pi}{2}]}$. They then asked whether this was a minimal set of relations. It is obvious that perpendicularity is not definable from intersection. (In fact, intersection alone suffices to formalize affine geometry.) So this amounted to asking whether intersection was definable from perpendicularity in $\mathbf{L}_1(\mathbb{R}^3)$.

This question was answered negatively in [1], by exhibiting an automorphism of $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet \rangle$ that does not preserve $\times^\bullet$. (The automorphisms of this "perpendicular geometry" do *not* preserve the points.)

So, if intersection is *not* definable in perpendicular geometry, which geometric relations *are* definable? The relation of parallel is easily seen to be definable. There is a quaternary relation $\equiv$, essentially an equivalence relation on unordered pairs of lines. For $A, B, C, D \in \mathbf{L}_1(\mathbb{R}^3)$, we say that $AB \equiv CD$ iff there is a rigid transformation σ (ie: induced by a composition of rotations and translations on $\mathbb{R}^3$) such that $\sigma(A) = C$ and $\sigma(B) = D$. The relations $\perp^\bullet$ and $\equiv$ are, in fact, inter-definable, that is, each is definable from the other. Furthermore, using these two relations, one can define "algebraic operations" on the equivalence classes of $\equiv$ that have "derivation-like" behavior. One can use this behavior to define the relations $\times^\bullet_\theta$ and $\times^\circ_\theta$ for any angle θ such that $\cos\theta$ is an algebraic number. In fact, we have the following theorem, which determines precisely the binary geometric relations that are definable from perpendicularity.

THEOREM 1. *Let $\boldsymbol{R} = \times^\bullet_\mathcal{A} \cup \times^\circ_\mathcal{B}$ with $\mathcal{A}, \mathcal{B} \subseteq [0, \frac{\pi}{2}]$ be an arbitrary geometric relation in $\boldsymbol{L}_1(\mathbb{R}^3)$. Then $\boldsymbol{R}$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \perp^\bullet \rangle$ iff $(\mathcal{A} \oplus \mathcal{B}) \sim \{0\}$ is a finite, nonempty set of angles, all of which have algebraic cosines, and $\mathcal{A}, \mathcal{B}$ are* *both finite unions of intervals and isolated points, such that the endpoints of the intervals and the isolated points all have algebraic cosines.*

This theorem has a converse, in which the algebraicity of the angles is not needed.

THEOREM 2. *Let $\mathcal{A}, \mathcal{B} \subseteq [0, \frac{\pi}{2}]$ with $(\mathcal{A} \oplus \mathcal{B}) \sim \{0\}$ a finite, nonempty set of angles. Then the relation $\perp^\bullet$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}} \rangle$.*

Thus, all of the binary relations definable in $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet \rangle$ are actually inter-definable with each other.

Now, suppose that we are given an arbitrary binary geometric relation $\mathbf{R}$ in $\mathbf{L}_1(\mathbb{R}^3)$. Ideally, we seek to determine which binary relations can be defined in $\langle \mathbf{L}_1(\mathbb{R}^3), \mathbf{R} \rangle$. In all of these cases considered so far, the essential (and easy!) first step has been to define the parallel relation $\| = \times^\bullet_0 \cup \times^\circ_0$. It turns out that the task of defining parallel is not so easy in general, if you don't know what you are trying to define it from. And all that we know now is that $\mathbf{R} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subseteq [0, \frac{\pi}{2}]$. Nevertheless, we have the following theorem. (We call $\mathbf{R}$ a *logical relation* provided that $\mathbf{R}$ is definable from equality alone. Otherwise, $\mathbf{R}$ is called *non-logical.* There are exactly four logical binary relations.)

THEOREM 3. *Let $\boldsymbol{R}$ be a non-logical geometric binary relation. Then $\|$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R} \rangle$.*

This theorem can be generalized. Let $\mathbf{L}_s(\mathbb{R}^n)$ with $0 \leq s \leq n$ be the set of all s-flats in $\mathbb{R}^n$. Again, the group of similarity transformations on $\mathbb{R}^n$ acts on $\mathbf{L}(\mathbb{R}^n)$, and we have a corresponding notion of a geometric relation on $\mathbf{L}(\mathbb{R}^n)$. The binary geometric relations have a similar description as $\mathbf{R} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}}$ where $\mathcal{A}$ and $\mathcal{B}$ are now sequences of angles of length s satisfying the restrictions $\mathcal{A} \subseteq \{\alpha \in \mathbb{R}^s \mid 0 \leq \alpha_1 \leq \cdots \leq \alpha_s \leq \frac{\pi}{2} \text{ with } \alpha_\ell = 0 \text{ for } \ell \leq 2s - n\}$ and $\mathcal{B} \subseteq \{\beta \in \mathbb{R}^s \mid 0 \leq \beta_1 \leq \cdots \leq \beta_s \leq \frac{\pi}{2} \text{ with } \beta_\ell = 0 \text{ for } \ell \leq 2s + 1 - n\}$. In this more general situation, we must be somewhat careful in stating the theorem, due to the existence of non-logical geometric equivalence relations on $\mathbf{L}_s(\mathbb{R}^n)$ other than $\|$.

For the record, we should note just exactly what the geometric equivalence relations on $\mathbf{L}_s(\mathbb{R}^n)$ are. Besides the logical equivalence relations and the parallel relation $\|$, the geometric equivalence relations are as follows.

For $s = 1$ and $n = 2$, there is a geometric equivalence relation for every subgroup $\mathbb{G}$ of $\mathbb{R}$ with $\pi \in \mathbb{G}$. The geometric equivalence relation corresponding to $\mathbb{G}$ is $\mathbf{R} = \times^\bullet_{\mathcal{A}}$ for $\mathcal{A} = \{\theta \in \mathbb{G} \mid 0 \leq \theta \leq \frac{\pi}{2}\}$.

For $n > 2$, there are no non-logical equivalence relations other than $\|$, unless $n = 2s$, in which case there is exactly one such relation. In this case, the relation $\mathbf{R} = \| \cup \times^\bullet_{\langle \frac{\pi}{2}, \ldots, \frac{\pi}{2} \rangle}$ is a geometric equivalence relation.

Given a geometric binary relation $\mathbf{R}$, we will refer to the relations $\mathbf{R}$, the compliment of $\mathbf{R}$, and the symmetric difference of the identity relation with either $\mathbf{R}$ or its compliment, as the *associates* of $\mathbf{R}$. Note that any relation is trivially inter-definable with any of its associates.

Having established the terminology, we can now state the main theorem on the definability of parallel.

Theorem 4. *Let $\boldsymbol{R}$ be an geometric binary relation on $\boldsymbol{L}_s(\mathbb{R}^n)$. Then $\|$ is definable in $\langle \boldsymbol{L}_s(\mathbb{R}^n, \boldsymbol{R})\rangle$ iff $\boldsymbol{R}$ is not an associate of a geometric equivalence relation other than parallel.*

This suggests an open problem, which we state in its simplest case, in the form of a conjecture.

Conjecture 1. *Let $\boldsymbol{R}$ be a non-logical* ternary *geometric relation on $\boldsymbol{L}_1(\mathbb{R}^3)$. Then $\|$ is definable from $\left\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R}\right\rangle$.*

Suppose that $\mathbf{R}$ is a binary geometric relation on $\mathbf{L}_s(\mathbb{R}^n)$ such that $\|$ is a congruence relation of the structure $\langle \mathbf{L}_s(\mathbb{R}^n), \mathbf{R}\rangle$. Such a relation, and its associates, we call *projective*. (If $\mathbf{R}$ is an associate of such a relation, we may replace $\mathbf{R}$ by its symmetric difference with the identity relation, if necessary.) In that case, if parallel is definable (which according to the above theorem, it is in all interesting cases), we may as well mod out by $\|$ and consider $\mathbf{R}$ as a geometric relation on $\mathbf{L}_{s-1}(\mathbb{P}(\mathbb{R}^n))$. Note here that the group action induced by the similarity transformations can be considered as the orthogonal group action on $\mathbb{P}(\mathbb{R}^n)$, the real $n-1$-dimensional projective space. (Thus, for us a projective geometric relation on $\mathbf{L}_{s-1}(\mathbb{P}(\mathbb{R}^n))$ is one that is preserved by all *metric preserving* bijections on the projective space $\mathbb{P}(\mathbb{R}^n)$.) Essentially, projective relations are those relations that are unable to distinguish between skew and intersecting pairs of s-flats of the same angle sequences, except possibly for parallel s-flats.

Theorem 5. *Let $\boldsymbol{R}$ be a non-projective binary geometric relation on $\boldsymbol{L}_s(\mathbb{R}^n)$. Then the ternary relation that asserts that three s-flats are mutually parallel and all lie in some $s+1$-flat is definable in $\langle \boldsymbol{L}_s(\mathbb{R}^n), \boldsymbol{R}\rangle$.*

Let us now return to the case of $s = 1$ and $n = 3$. The following theorem simply states that the set of all binary relations definable in $\left\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet, \times^\bullet\right\rangle$ is correct, given that we already know that the two binary relations of perpendicularity and intersection suffice to formalize 3-dimensional Euclidean geometry.

Theorem 6. *Let $\boldsymbol{R} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}}$ with $\mathcal{A}, \mathcal{B} \in [0, \frac{\pi}{2}]$ be a binary geometric relation on $\boldsymbol{L}_1(\mathbb{R}^3)$. Then $\boldsymbol{R}$ is definable in $\left\langle \boldsymbol{L}_1(\mathbb{R}^3), \perp^\bullet, \times^\bullet\right\rangle$ iff $\mathcal{A}$ and $\mathcal{B}$ can both be written as finite unions of intervals and isolated points so that the endpoints of the intervals and the isolated points all have algebraic cosines.*

Theorem 7. *Let $\boldsymbol{R} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}}$ with $\mathcal{A}, \mathcal{B} \in [0, \frac{\pi}{2}]$ be a binary geometric relation on $\boldsymbol{L}_1(\mathbb{R}^3)$. If $\mathcal{A} \oplus \mathcal{B}$ contains a nonempty open interval, then $\times^\bullet$ is definable in $\left\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R}\right\rangle$. Furthermore, if in addition $(0, \frac{\pi}{2}] \not\subseteq \mathcal{A} \oplus \mathcal{B}$ (ie: $\boldsymbol{R}$ is* not *an associate of $\times^\bullet$), and if $\mathcal{A} \oplus \mathcal{B}$ can be written as a finite union of intervals and isolated points, then $\perp^\bullet$ is also definable in $\left\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R}\right\rangle$. (Note that in the latter case, if the endpoints of the intervals and the isolated points all have algebraic cosines, then $\boldsymbol{R}$ is a single binary geometric relation that suffices to formalize 3-dimensional Euclidean geometry.)*

Now, let us consider a projective binary geometric relation $\mathbf{R}$ on $\mathbf{L}_1(\mathbb{R}^3)$. By replacing $\mathbf{R}$ with its symmetric difference with the identity relation, if necessary, we may assume that $\mathbf{R} = \times_{\mathcal{A}} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{A}}$ for some $\mathcal{A} \subseteq [0, \frac{\pi}{2}]$. As mentioned above, since $\mathbf{R}$ cannot distinguish between parallel lines in $\mathbb{R}^3$, we may as well mod out by $\|$ and think of $\mathbf{R}$ as a relation between points in the projective plane $\mathbb{P}(\mathbb{R}^3)$, two points being $\mathbf{R}$ related provided that the distance between the two points in the projective plane is in $\mathcal{A}$. For any $\theta \in [0, \frac{\pi}{2}]$, we define $\times_\theta = \times^\bullet_\theta \cup \times^\circ_\theta$, and $\perp = \times_{\frac{\pi}{2}}$.

THEOREM 8. *Let $\boldsymbol{R} = \times_{\mathcal{A}}$ with $\mathcal{A} \subseteq [0, \frac{\pi}{2}]$ be a projective binary geometric relation. Then $\boldsymbol{R}$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \perp \rangle$ iff $\mathcal{A}$ can be written as a finite union of intervals and isolated points where the cosines of the endpoints of the intervals and the isolated points are all algebraic.*

We conjecture the converse, without the algebraicity assumption.

CONJECTURE 2. *Let $\boldsymbol{R} = \times_{\mathcal{A}}$ with $\mathcal{A} \subseteq [0, \frac{\pi}{2}]$ be a projective binary geometric relation. Suppose that $\mathcal{A}$ can be written as a finite union of intervals and isolated points. Then $\perp$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R} \rangle$.*

We can prove some special cases of this conjecture.

THEOREM 9. *Let $\boldsymbol{R} = \times_{\mathcal{A}}$ with $\mathcal{A} \subseteq [0, \frac{\pi}{2}]$ be a projective binary geometric relation. Suppose that $\mathcal{A}$ can be written as a finite union of intervals and isolated points. Suppose further that one of the following holds:*
1. *$\mathcal{A} = [0, \theta)$ for some θ with $0 < \theta \leq \frac{\pi}{2}$.*
2. *$\mathcal{A} = [0, \theta]$ for some θ with $0 < \theta < \frac{\pi}{2}$.*
3. *$\mathcal{A} \cap [\frac{\pi}{4}, \frac{\pi}{2}]$ is either finite, or cofinite in $[\frac{\pi}{4}, \frac{\pi}{2}]$.*

Then $\perp$ is definable in $\langle \boldsymbol{L}_1(\mathbb{R}^3), \boldsymbol{R} \rangle$.

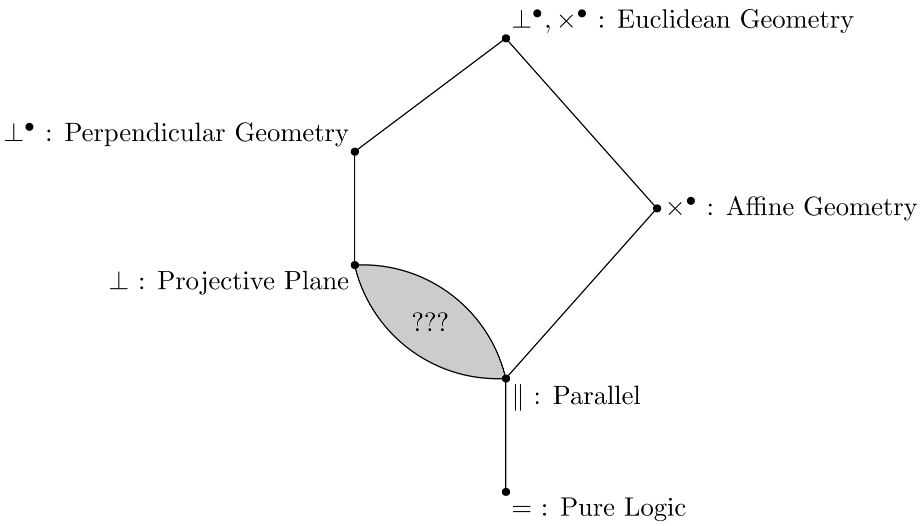

FIGURE 1. Definability structure for elementary Euclidean binary relations

Let us call a geometric relation $\mathbf{R}$ on $\mathbf{L}_1(\mathbb{R}^3)$ an elementary Euclidean relation provided that $\mathbf{R}$ is definable in $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet, \times^\bullet \rangle$. This is equivalent to saying that $\mathbf{R} = \times^\bullet_{\mathcal{A}} \cup \times^\circ_{\mathcal{B}}$ where $\mathcal{A}, \mathcal{B} \subseteq [0, \frac{\pi}{2}]$ can both be written as a finite union of intervals and isolated points such that the endpoints of the intervals and the isolated points all have algebraic cosines.

The above theorems then classify the elementary Euclidean binary relations into the geometries shown in Figure 1, modulo the mysteries of the grey area of the definability structure of the projective plane, which is conjectured to be empty.

If the conjecture is true, then the only new geometry in Figure 1 is perpendicular geometry, that is, the structure $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet \rangle$. So, let us look a bit closer at this structure. An examination of the automorphism group of this structure reveals some rather curious and amusing algebraic results and conjectures, some of which, if true, surely must have some geometry lurking in the background as the ultimate cause.

First of all, just as the automorphism group of 3-dimensional Euclidian geometry (which we now identify with the structure $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet, \times^\bullet \rangle$) is generated by rotations, reflections, dilations and translations, so is the automorphism group of perpendicular geometry $\langle \mathbf{L}_1(\mathbb{R}^3), \perp^\bullet \rangle$ generated by rotations, reflections, dilations and "generalized translations".

Let $\mathbb{S}^2$ be the set of unit vectors in $\mathbb{R}^3$. In the case of perpendicular geometry, a "generalized translation" reduces to a map $\delta : \mathbb{S}^2 \to \mathbb{R}^3$, which satisfies $\delta(\mathbf{a} \times \mathbf{b}) = \delta(\mathbf{a}) \times \mathbf{b} + \mathbf{a} \times \delta(\mathbf{b})$ whenever $\mathbf{a} \cdot \mathbf{b} = 0$ with $\|a\| = \|b\| = 1$. Any such map defines a "generalized translation" via the map that takes that line A, and maps it to $\sigma(A) = A + \mathbf{a} \times \delta(\mathbf{a})$, where $\mathbf{a}$ is a any unit direction vector for the line A.

It turns out, for geometric reasons, that any such map must extend uniquely to a derivation on the cross product ring. Related to this is the following theorem.

Theorem 10. *Let $\langle \mathbb{R}^3, \mathbf{0}, +, -, \times \rangle$ be the cross product ring of 3-dimensional vectors. Suppose that $\delta : \mathbb{R}^3 \to \mathbb{R}^3$ is a map that satisfies the product rule*

$$\delta(\mathbf{x} \times \mathbf{y}) = \delta(\mathbf{x}) \times \mathbf{y} + \mathbf{x} \times \delta(\mathbf{y}) \textit{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

Then δ also satisfies the sum rule

$$\delta(\mathbf{x} + \mathbf{y}) = \delta(\mathbf{x}) + \delta(\mathbf{y}) \textit{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

Furthermore, there exists a unique field derivation $\delta_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\delta_0(\mathbf{x} \cdot \mathbf{y}) = \delta(\mathbf{x}) \cdot \mathbf{y} + \mathbf{x} \cdot \delta(\mathbf{y}) \textit{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$$

and

$$\delta(\lambda \mathbf{x}) = \delta_0(\lambda)\mathbf{x} + \lambda \delta(\mathbf{x}) \textit{ for all } \mathbf{x} \in \mathbb{R}^3 \textit{ and } \lambda \in \mathbb{R}.$$

That is, any map on the cross product ring in $\mathbb{R}^3$ that satisfies the product rule also satisfies the sum rule. Indeed, not only do we get the sum rule for free, we also get a uniquely determined scalar derivation and the product rules for the dot product between vectors, and for the scalar product between scalars and vectors are also free.

It is natural to make the following conjecture.

Conjecture 3. *Let L be a finite dimensional semi-simple Lie algebra over the complex numbers $\mathbb{C}$. Suppose that $\delta : L \to L$ satisfies the product rule for the Lie bracket $\delta[x, y] = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in L$. Then L also satisfies the sum rule $\delta(x + y) = \delta(x) + \delta(y)$ for all $x, y \in L$.*

(Edited to add: This conjecture is now known to be true.)
This conjecture is known to be valid for the smallest case, that of the 3-dimensional simple Lie algebra over $\mathbb{C}$. Now, suppose that δ satisfies both the hypothesis and the conclusion of Conjecture 3. If the previous theorem about the cross product ring is any guide, we ought to get a scalar derivation for free, together with the appropriate product rules for scalar multiplication and the Killing form. In fact, this is true in the simple case, and in the semi-simple case we get a scalar derivation for every simple factor. The next theorem says that things are as nice as can possibly be hoped for. Of course, since the above Conjecture is not yet a Theorem, we will have to assume the sum rule, in addition to the product rule, at least for now.

THEOREM 11. *Let L be a finite dimensional semi-simple Lie Algebra over $\mathbb{C}$. Suppose that $\delta : L \to L$ satisfies both the product rule and the sum rule (ie: δ is a derivation on L, as a Lie* ring, *thus making δ linear over $\mathbb{Q}$.) Let $L = \bigoplus_{i=1}^n L_i$ be the decomposition of L into its simple factors, and for any $x \in L$, we write $x = \sum_{i=1}^n x_i$ with $x_i \in L_i$ for each i. Then there exists a unique sequence of maps $\delta_i : \mathbb{C} \to \mathbb{C}$ such that*

1. *δ_i is a derivation on $\mathbb{C}$ for every i.*
2. *$\delta(\lambda x) = (\sum_{i=1}^n \delta_i(\lambda) x_i) + \lambda\delta(x)$ for every $x \in L$ and $\lambda \in \mathbb{C}$.*
3. *$\sum_{i=1}^n \delta_i \langle x_i, y_i \rangle = \langle \delta(x), y \rangle + \langle x, \delta(y) \rangle$ for every $x, y \in L$.*

If R is a ring (not necessarily commutative nor associative), a map $\delta : R \to R$ is called a *production* if it satisfies the product rule. We can easily classify all productions on $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. In the case of $\mathbb{Z}$ and $\mathbb{Q}$, any function from the set of primes to the ring extends uniquely to a production. For $\mathbb{R}$ and $\mathbb{C}$, any function from $\{e^h \mid h \in H\}$ to the field extends uniquely to production, provided that H is a Hammel basis. In all cases, productions are rarely additive.

Regarding the Quaternions $\mathbb{H}$, we have the following characterization of productions.

THEOREM 12. *Let $\delta' : \mathbb{R} \to \mathbb{R}$ be a production, and let $\delta_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be a production (and thus a derivation) on the cross product ring $\left\langle \mathbb{R}^3, \mathbf{0}, +, -, \times \right\rangle$, with $\delta_0 : \mathbb{R} \to \mathbb{R}$ as its corresponding scalar derivation. Identify $\mathbb{R}$ with the real part of $\mathbb{H}$ and $\mathbb{R}^3$ with the "vector part" of $\mathbb{H}$. Then, the following map $\delta : \mathbb{H} \to \mathbb{H}$ is a production, where $x = x_0 + \mathbf{x}_1$ is the decomposition of the x into its real and vector components, and where $\|x\|$ is the standard norm of x in $\mathbf{H}$.*

$$\begin{aligned} \delta(x) &= \delta'(\|x\|)\frac{x}{\|x\|} + \|x\|\left(\delta_0\left(\frac{x_0}{\|x\|}\right) + \delta_1\left(\frac{\mathbf{x}_1}{\|x\|}\right)\right) \\ &= \big(\delta'(\|x\|) - \delta_0(\|x\|)\big)\frac{x}{\|x\|} + \delta_0(x_0) + \delta_1(\mathbf{x}_1) \text{ for all } x \neq 0 \end{aligned}$$

(Of course, we always have $\delta(0) = 0$.) Furthermore, every production δ on $\mathbb{H}$ can be written uniquely in this form.

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