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# THE UNDEFINABILITY OF INTERSECTION FROM PERPENDICULARITY IN THE THREE-DIMENSIONAL EUCLIDEAN GEOMETRY OF LINES

ABSTRACT. The undefinability in question is proved by constructing a bijection of lines preserving perpendicularity but not intersection.

## INTRODUCTION

In Schwabhäuser and Szczerba [3] formalizations of Euclidean geometry are considered in which the universe consists of lines and the primitive notions are relations on lines. A set of primitive notions sufficient to formalize $n$-dimensional Euclidean geometry is found for each $n \geqslant 2$. For $n \geqslant 4$, the single binary notion of perpendicularity (two lines intersecting at a right angle) suffices. For $n = 2$, this notion in conjunction with the ternary notion of copunctuality (three lines intersecting at a single point) suffices and is shown to be minimal, in the sense that neither notion is definable from the other. For the remaining case of $n = 3$, the two binary notions of perpendicularity and intersection are shown to be sufficient. It is easy to see that perpendicularity is not definable from intersection. (The proof uses Padoa's method.) Schwabhäuser and Szczerba ask whether intersection is definable from perpendicularity for $n = 3$. This paper answers that negatively, thus proving that perpendicularity and intersection form a minimal set of primitive notions sufficient to formalize the three-dimensional Euclidean geometry of lines.

## PRELIMINARIES

$\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}^3$ is the three-dimensional vector space over $\mathbb{R}$. The operations $\cdot$ and $\times$ are the dot and cross products, respectively. $|a|$ is the length of the vector $a$. In this paper all vectors are in $\mathbb{R}^3$.

A line is any set $A = \{p + \lambda a \mid \lambda \in \mathbb{R}\}$, where $p$ and $a$ are vectors. If $|a| = 1$, then $a$ is called a direction vector of $A$. Every line has exactly two direction vectors. If $a$ is a direction vector, then so is $-a$. Let $\mathscr{L}$ be the set of all lines.

Throughout this paper, real numbers will be denoted by lowercase Greek letters (except $\delta$ and $\theta$), vectors by lowercase Roman letters, and lines by uppercase Roman letters. Given a line denoted by an uppercase Roman

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letter, the corresponding lowercase letter will denote one of its direction vectors.

One vector identity which will be used is the following:

(1) $$a \times b \cdot c \times d = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

The validity of this is easily checked.

Let $p$ lie on $A$ and $q$ lie on $B$, where $A$ and $B$ are not parallel. It is easy to check that

(2) $A$ intersects $B$ iff $(p - q) \cdot a \times b = 0.$

A function $\delta\colon \mathbb{R} \to \mathbb{R}$ is called a derivation provided $\delta(\alpha + \beta) = \delta(\alpha) + \delta(\beta)$ and $\delta(\alpha\beta) = \delta(\alpha)\beta + \alpha\delta(\beta)$ for any $\alpha$ and $\beta$. Note that $\delta$ is *not* required to be linear over $\mathbb{R}$. If $\delta$ is a derivation and $\alpha$ is algebraic, then $\delta(\alpha) = 0$. (Thus, the product rule implies that $\delta$ is linear over the algebraic reals.) The existence of nontrivial derivations is guaranteed by the following fact.

FACT. *Let $\alpha$ be any transcendental number. Then there exists a derivation $\delta$ with $\delta(\alpha) \neq 0$.*

This is a special case of Lang [2, p. 267].

Given a derivation $\delta$, we define a function $\theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\bar{\delta}(\langle \alpha, \beta, \gamma \rangle) = \langle \delta(\alpha), \delta(\beta), \delta(\gamma) \rangle$. $\bar{\delta}$ satisfies many identities similar to vector calculus identities. Among these are

(3) $$\bar{\delta}(-a) = -\bar{\delta}(a),$$

(4) $$\delta(a \cdot b) = \bar{\delta}(a) \cdot b + a \cdot \bar{\delta}(b),$$

(5) $$|a| = 1 \quad \text{implies} \quad a \cdot \bar{\delta}(a) = 0,$$

and

(6) $$|a| = 1 \quad \text{implies} \quad a \times \bar{\delta}(a) \cdot a \times b = \bar{\delta}(a) \cdot b.$$

The proofs of (3) and (4) are analogous to the proofs of the corresponding vector calculus identities. For (5), we have

$$\begin{aligned} a \cdot \bar{\delta}(a) &= \frac{1}{2}(\bar{\delta}(a) \cdot a + a \cdot \bar{\delta}(a)) \\ &= \frac{1}{2}\delta(a \cdot a) \\ &= \frac{1}{2}\delta(1) \\ &= 0. \end{aligned}$$

The proof of (6) uses (1) and (5).

$$a \times \bar{\delta}(a) \cdot a \times b = (a \cdot a)(\bar{\delta}(a) \cdot b) - (a \cdot b)(\bar{\delta}(a) \cdot a)$$
$$= \bar{\delta}(a) \cdot b$$

## THE UNDEFINABILITY OF INTERSECTION

For any derivation $\delta$, we define a function $\theta_\delta \colon \mathscr{L} \to \mathscr{L}$ by

$$(7) \qquad \theta_\delta(A) = \{p + a \times \bar{\delta}(a) \mid p \in A\},$$

where $a$ is a direction vector of $A$. Using (3), we have $(-a) \times \bar{\delta}(-a) = a \times \bar{\delta}(a)$, so that the right side of (7) does not depend on which direction vector $a$ of $A$ is chosen. Thus $\theta_\delta$ is well defined. Clearly $\theta_\delta$ is a bijection on $\mathscr{L}$ which translates each line parallel to itself. Note that $a$ is a direction vector of $\theta_\delta(A)$.

THEOREM 1. *Let $\delta$ be a derivation and let $a$ and $b$ be direction vectors of $A$ and $B$, respectively. If $A$ intersects $B$, then $\theta_\delta(A)$ intersects $\theta_\delta(B)$ if and only if $\delta(a \cdot b) = 0$.*

*Proof.* Let $A$ intersect $B$. Then either $A = B$ or $A$ is not parallel to $B$. If $A = B$ the conclusion follows easily, since in that case $\delta(a \cdot b) = \delta(\pm 1) = 0$. Thus, we can assume that $A$ is not parallel to $B$. Let $p$ be the point of intersection of $A$ and $B$. By (7), $p + a \times \bar{\delta}(a)$ lies on $\theta_\delta(A)$ and $p + b \times \bar{\delta}(b)$ lies on $\theta_\delta(B)$. By (2), we have

$$\theta_\delta(A) \text{ intersects } \theta_\delta(B) \text{ iff } ((p + a \times \bar{\delta}(a)) - (p + b \times \bar{\delta}(b))) \cdot a \times b = 0.$$

But by (6) and (4),

$$((p + a \times \bar{\delta}(a)) - (p + b \times \bar{\delta}(b))) \cdot a \times b = a \times \bar{\delta}(a) \cdot a \times b + b \times \bar{\delta}(b) \cdot b \times a$$
$$= \bar{\delta}(a) \cdot b + \bar{\delta}(b) \cdot a$$
$$= \delta(a \cdot b).$$

The theorem follows. □

Since $\theta_\delta$ translates every line parallel to itself, it preserves perpendicularity if and only if it preserves the intersections of perpendicular lines. Therefore, the following corollary is an easy consequence of Theorem 1.

COROLLARY. *Let $\delta$ be a derivation. Then $\theta_\delta$ preserves perpendicularity. That is, if $A$ is perpendicular to $B$, then $\theta_\delta(A)$ is perpendicular to $\theta_\delta(B)$.*

The main result is stated in Theorem 2.

THEOREM 2. *Intersection is not definable from perpendicularity in the three-dimensional Euclidean geometry of lines.*

*Proof.* The proof uses Padoa's method (see Beth [1, pp. 87–89]). Let $\alpha$ be any transcendental number between $-1$ and 1. Then there exists a derivation $\delta$ with $\delta(\alpha) \neq 0$. By the corollary to Theorem 1, $\theta_\delta$ is a bijection of lines which preserves perpendicularity and thus preserves every relation on $\mathscr{L}$ definable from perpendicularity.

If we choose any unit vectors $a$ and $b$ such that $a \cdot b = \alpha$ and choose any pair of intersecting lines $A$ and $B$ with direction vectors $a$ and $b$, respectively, we see by Theorem 1 that $\theta_\delta(A)$ does not intersect $\theta_\delta(B)$, and hence $\theta_\delta$ does not preserve intersection. The theorem follows. □

## REFERENCES

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