LIE THEORY SEMINAR 3/10/2017 1

Weight modules over Noncommutative Kleinian fiber products

① Kleinian Singularities

$X = \mathbb{C}^2/\Gamma$, $\Gamma \subset SL(2,\mathbb{C})$ finite subgroup

Type A_n, $n \geq 1$

D_n, $n \geq 4$

E_6, E_7, E_8

$\mathcal{O}_X = \mathbb{C}[x,y,z]/(xy - z^{n+1})$ (type A_n)

② Noncommutative Kleinian Singularities

Hodges 1993 (Type A)

Crawley-Boevey 1998 (ADE)

$\hbar \in \mathbb{C}$ deformation parameter

$f \in \mathbb{C}[u]$ polynomial $Z(f) \subseteq \frac{\hbar}{2} + \mathbb{Z}\hbar$

$$\mathcal{A}_\hbar(f) = \mathbb{C}\langle X^+, X^-, H\rangle \Big/ \Big\langle \begin{array}{l} [H, X^\pm] = \pm\hbar X^\pm \\ X^\pm X^\mp = f(H \mp \frac{\hbar}{2}) \end{array} \Big\rangle$$

Properties.

1) $\mathcal{A}_0(f) \cong \mathcal{O}_X$, X of type $A_{\deg f - 1}$

2) $\mathcal{A}_\hbar(f)$ is a generalized Weyl algebra (Bavula)

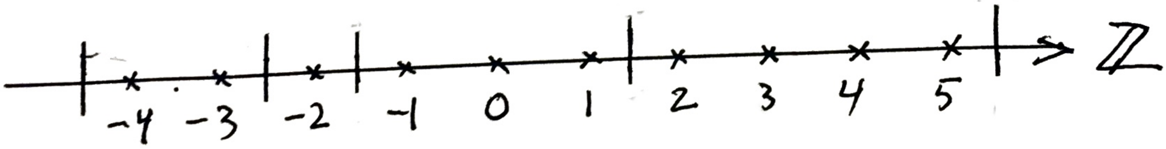
In fact $\mathcal{A}_1(u - \frac{1}{2}) \cong A_1(\mathbb{C})$ first Weyl algebra

$$X^+ \mapsto x$$
$$X^- \mapsto \partial$$
$$H \mapsto \partial x$$

3) $\mathcal{A}_1((u-\frac{1}{2})(u-\frac{1}{2}-d)) \cong \dfrac{U(\mathfrak{sl}_2)}{\operatorname{Ann} L(d-1)}$

3 Integral weight modules over $\mathcal{A}_1(f)$

$$M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda, \quad M_\lambda = \{v \in M \mid Hv = \lambda v\}$$
$$\operatorname{Supp}(M) = \{\lambda \in \mathbb{Z} \mid M_\lambda \neq 0\}$$



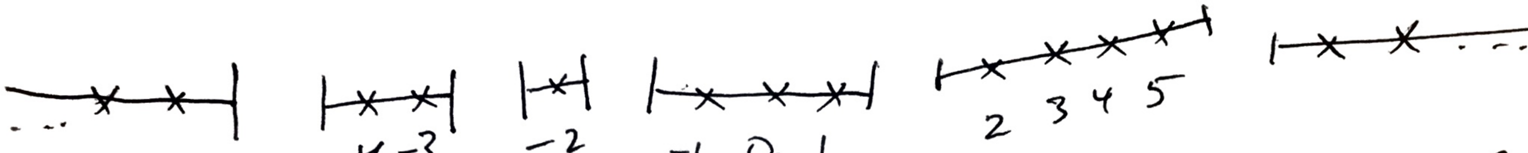
$|$ = zero of f, $\in \frac{1}{2} + \mathbb{Z}$

$\times$ = integer

Here $f(u) = (u + \frac{9}{2})(u + \frac{5}{2})(u + \frac{3}{2})(u - \frac{3}{2})(u - \frac{11}{2})$

Simple modules:

Cut $\mathbb{R}$ into pieces, D, one cut at every zero of f:



Then: There is a unique simple integral weight module $M(D)$ corresponding to each piece D, and $\mathrm{supp}(M(D)) = D \cap \mathbb{Z}$.

Every simple integral weight module occurs this way.

(Bavula, 1992, Drozd-Guzner-Ovsienko 1995)

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④ Fiber products of Kleinian Singularities

Product: $\mathcal{O}_{X_n \times X_m} \cong \mathcal{O}_{X_n} \underset{\mathbb{C}}{\otimes} \mathcal{O}_{X_m} = \dfrac{\mathbb{C}[x_1, y_1, z_1, x_2, y_2, z_2]}{(x_1y_1 - z_1^n, x_2y_2 - z_2^m)}$

Fiber product $X_n \underset{Z}{\times} X_m$, $Z = \{ z_1 = z_2 \}$

$$\mathcal{O}_{X_n \underset{Z}{\times} X_m} = \mathcal{O}_{X_n} \underset{\mathbb{C}[z]}{\otimes} \mathcal{O}_{X_m} = \frac{\mathbb{C}[x_1, y_1, x_2, y_2, z]}{(x_1y_1 - z^n, x_2y_2 - z^m)}$$

Noncomm. Deformation.

$(\alpha, \beta) \in \mathbb{C}^2$ deformation parameters
$= (\alpha_1, \alpha_2)$

$0 \neq P_1, P_2 \in \mathbb{C}[u]$ monic,

$$Z(P_i) \subseteq \frac{\alpha_i}{2} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 = E_i$$

$$\tilde{\mathcal{A}}_{\alpha,\beta}(P_1, P_2) = \frac{\mathbb{C}\langle X_1^+, X_1^-, X_2^+, X_2^-, H\rangle}{\left\langle \begin{array}{l} [H, X_i^\pm] = \pm\alpha_i X_i^\pm \\ [X_1^\pm, X_2^\mp] = 0 \\ X_i^\pm X_i^\mp = P_i(H \mp \frac{\alpha_i}{2}) \end{array} \right\rangle}$$

$$\mathcal{A}_{\alpha\beta}(P_1, P_2) = \tilde{\mathcal{A}}_{\alpha,\beta}(P_1, P_2) / \mathcal{I}$$

$$\mathcal{I} = \left\{ a \in \tilde{\mathcal{A}}_{\alpha\beta}(P_1, P_2) \;\middle|\; \begin{array}{l} f(H)\cdot a = 0 \text{ for some} \\ \text{nonzero } f \in \mathbb{C}[u] \end{array} \right\}$$

Thm TFAE

i) $\mathcal{A}_{\alpha\beta}(P_1, P_2) \neq \{0\}$

ii) $\exists!\ \mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ $\quad \mathcal{L}_i : E_i \to \mathbb{Z}_{\geq 0}$

$|\mathcal{L}_i^{-1}([1, \infty))| < \infty$, $\quad \forall v \in V = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \frac{\alpha_1 + \alpha_2}{2}$

$$\mathcal{L}_1(v + \alpha_2) + \mathcal{L}_2(v + \alpha_1) = \mathcal{L}_1(v - \alpha_2) + \mathcal{L}_2(v - \alpha_1)$$

such that

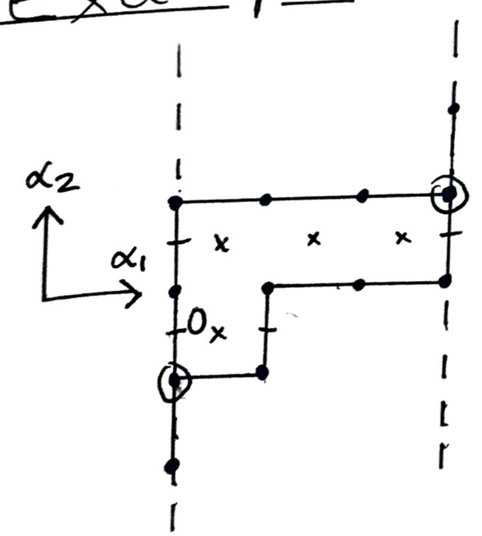
$$p_i(u) = P_i^{\mathcal{L}}(u) = \prod_{e \in E_i} (u-e)^{\mathcal{L}_i(e)}, \quad i = 1, 2.$$

Moreover, if $\mathcal{L} \neq 0$, $\exists!\,(m,n) \in \mathbb{Z}_{>0}$, $\gcd(m,n) = 1$ such that $m\alpha_1 + n\alpha_2 = 0$.

After rescaling WLOG $(\alpha_1, \alpha_2) = (-n, m)$.

Notation: $\mathcal{A}(\mathcal{L}) = \mathcal{A}_{-n,m}(p_1^{\mathcal{L}}, p_2^{\mathcal{L}})$.

Example $(m,n) = (3,2) \Rightarrow (\alpha_1, \alpha_2) = (-2, 3)$



$$F = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 \cong \frac{\mathbb{Z}^2}{\mathbb{Z}\cdot(m,n)} = \frac{\mathbb{Z}^2}{\mathbb{Z}\cdot(3,2)}$$

face lattice

$$E_i = F + \frac{\alpha_i}{2}$$ midpoints of vertical, horizontal edges

$$V = F + \frac{\alpha_1 + \alpha_2}{2}$$ vertex lattice

$$\simeq \frac{(\mathbb{Z} + \frac{1}{2})^2}{\mathbb{Z}\cdot(m,n)}$$

vertical edges:

$$P_1^{\mathcal{L}}(u) = \left(u - \left(\frac{-\alpha_1}{2}\right)\right)\left(u - \frac{\alpha_1}{2}\right)\left(u - \left(\frac{-\alpha_1}{2} + \alpha_2\right)\right)\left(u - \left(3\alpha_1 - \frac{\alpha_1}{2} + \alpha_2\right)\right)$$

$$= (u-1)(u+1)(u-4)(u+2)$$

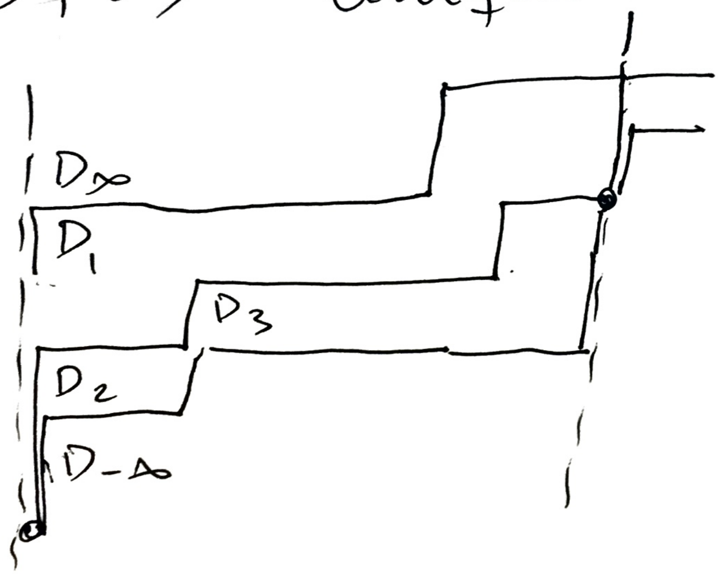
Similarly for $P_2^{\mathcal{L}}(u)$

Thm Simple weight integral $\mathcal{A}(\mathcal{L})$ - modules correspond to conn. components D of $\mathbb{R}^2/\mathbb{Z}\cdot(m,n) \setminus \overline{\mathcal{L}}$. If $\pi_1(D)=\mathbb{Z}$ then

$M(D, \xi)$, $\xi^\times$ 1-param. family

If $\pi_1(D) = 1$ then

$M(D, 0)$ unique



$M(D_1, \xi)$