Unitarizable Representations.

(J.T.H.)

$$\pi : G \to GL(V)$$

Suppose V has $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$

(i) $\langle \lambda u + \mu v, w \rangle = \lambda \langle u, w \rangle + \mu \langle v, w \rangle$

(ii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(iii) $u \neq 0 \Rightarrow \langle u, u \rangle > 0$

Call $\langle \cdot, \cdot \rangle$ an <u>inner product</u> on V.

The <u>contragredient rep</u> is

$$\pi^* : G \to GL(V)$$

$$\pi^*(g) = \pi(g^{-1})^*$$

adjoint operator
$\langle Av, w \rangle = \langle v, A^* w \rangle$

<u>Def</u> π is <u>unitary</u> if $\pi^* = \pi$

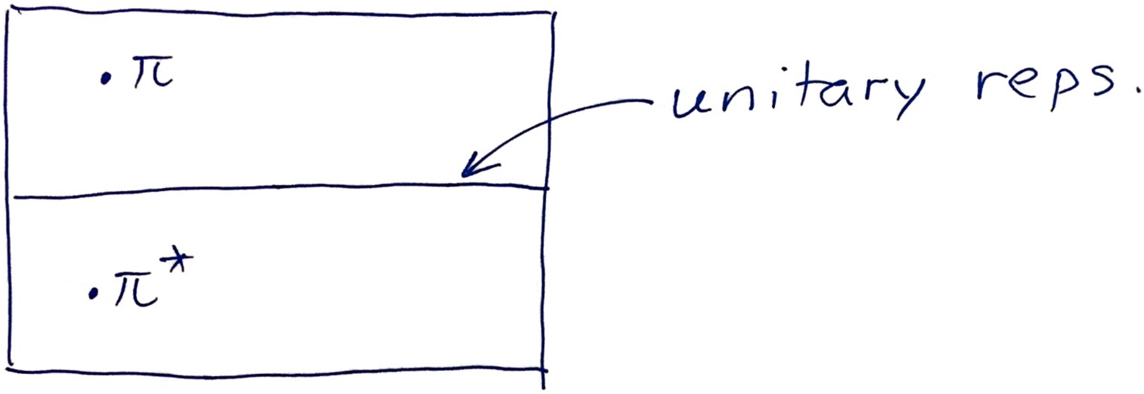
This means $\pi(g^{-1})^* = \pi(g) \quad \forall g \in G$

$\iff \pi(g)^{-1} = \pi(g)^* \quad \forall g \in G$

$\iff \pi(G) \subseteq U(V, \langle \cdot, \cdot \rangle)$

unitary group

Space of all reps of G:



<u>Def</u> π is <u>unitarizable</u> if π is equivalent to a unitary rep.

<u>Lem</u> $\pi: G \to GL(W)$ is unitarizable iff $\exists$ inner product $(\cdot,\cdot)$ on W such that $\pi(g)^* = \pi(g^{-1})$ wrt $(\cdot,\cdot)$.

<u>pf</u> If $(\cdot,\cdot)$ exists π is already unitary. Conversely suppose π is equivalent to a unitary rep

$$\rho: G \to GL(V), \quad (V, \langle \cdot,\cdot \rangle)$$

That means $\exists T: W \to V$ linear isomorphism such that

$$T \circ \pi(g) = \rho(g) \circ T \quad \forall g \in G$$

Define $(\cdot,\cdot) : W \times W \to \mathbb{C}$ by

$$(w_1, w_2) = \langle T(w_1), T(w_2) \rangle$$

Can check $(\cdot,\cdot)$ is an inner product.

Moreover

$$(\pi(g)(w_1), w_2) = \langle T \circ \pi(g)(w_1), T(w_2) \rangle =$$

$$= \langle \rho(g) \circ T(w_1), T(w_2) \rangle =$$

$$= \langle T(w_1), \rho(g)^* \circ T(w_2) \rangle$$

$$= \langle T(w_1), \rho(g^{-1}) \circ T(w_2) \rangle$$

$$= \langle T(w_1), T \circ \pi(g^{-1})(w_2) \rangle$$

$$= (w_1, \pi(g^{-1}) w_2)$$

proving $\pi(g)^* = \pi(g^{-1})$ wrt $(\cdot,\cdot)$.

QED

Thm If G is a compact topological group (for example a finite group) then any finite-dimensional representation of G is unitarizable.

Proof Any cpt top grp G has a unique measure dg, called the Haar measure, such that

(i) dg is right invariant

$$\int_G f(gh)\,dg = \int_G f(g)\,dg \quad \forall f \in L^1(G, dg),\ h \in G$$

(ii) $$\int_G dg = 1$$

Ex G finite $$\int_G f(g)\,dg = \frac{1}{|G|}\sum_{g\in G} f(g)$$

Ex $G = S^1 \cong \mathbb{R}/\mathbb{Z}$ $$\int_G f(g)\,dg = \int_0^1 f(x)\,dx$$

Now let $\pi: G \to GL(V)$ be any f.d. rep of G.

Pick any basis $\{v_i\}$ for V and define $\langle \cdot , \cdot \rangle$ by $\langle v_i, v_j \rangle = \delta_{ij}$.

(i.e. pick any inner product $\langle \cdot,\cdot \rangle$ on V)

Define a new inner product $(\cdot , \cdot)$ on V by

$$(u, v) = \int_G \langle \pi(g)u, \pi(g)v \rangle \, dg$$

Then one can check $(\cdot , \cdot)$ is an inner product and $\forall h \in G, \forall u, v \in V$:

$$(\pi(h)u, v) = \int_G \langle \pi(g)\pi(h)u, \pi(g)v \rangle \, dg =$$

$$= \int_G \langle \pi(gh)u, \pi(g)v \rangle \, dg = \left[g \mapsto gh^{-1} \right] =$$

$$= \int_G \langle \pi(g)u, \pi(gh^{-1})v \rangle \, dg =$$

$$= \int_G \langle \pi(g)u, \pi(g)\pi(h^{-1})v \rangle \, dg =$$

$= (u, \pi(h^{-1})v)$

which proves that $\pi(h)^* = \pi(h^{-1})$

Hence π is unitarizable. Q.E.D.

<u>Def</u> A <u>$*$-algebra</u> $\mathcal{A}$ is an associative unital $\mathbb{C}$-algebra equipped with an <u>involution</u> $*: \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$

$$(a+b)^* = a^* + b^* \qquad (\lambda a)^* = \bar{\lambda}(a^*)$$
$$(ab)^* = b^* a^* \qquad (a^*)^* = a$$

<u>Ex</u> G finite grp

The group algebra $\mathbb{C}G$ is a $*$-algebra with respect to

$g^* = g^{-1}$ $\quad \left(\text{i.e.} \left(\sum_{g \in G} \lambda_g g\right)^* = \sum_{g \in G} \bar{\lambda}_g g^{-1}\right)$

<u>Ex</u> $L^1(G, dg)$ is a $*$-alg

$$f^*(g) = \overline{f(g)}$$

Ex The 1st Weyl algebra

$$A_1(\mathbb{C}) = \frac{\mathbb{C}\langle X, \partial \rangle}{\langle \partial X - X\partial - 1 \rangle}$$

$A_1(\mathbb{C})$ is a $*$-algebra w.r.t.

$$X^* = \partial \quad , \quad \partial^* = X$$

Def A rep $\pi : \mathcal{A} \to \mathrm{End}(V)$ of a $*$-algebra is *unitarizable* if V has an inner product $\langle \cdot , \cdot \rangle$ such that

(1) $\langle \pi(a)u, v \rangle = \langle u, \pi(a^*)v \rangle \quad \forall a \in \mathcal{A}, u, v \in V.$

Ex $A_1(\mathbb{C})$ has a rep

$$\pi : A_1(\mathbb{C}) \to \mathrm{End}(\mathbb{C}[x])$$

$$\pi(X) = (p(x) \longmapsto x\,p(x))$$

$$\pi(\partial) = \left(p(x) \longmapsto \frac{d}{dx} p(x)\right)$$

Claim: π is unitarizable.

In fact there is a unique innerproduct satisfying (1) above and such that $\langle 1, 1 \rangle = 1$.

First we show $\langle x^n, x^m\rangle = 0$ if $m \neq n$:

Say $n > m$. Then

$$\langle x^n, x^m\rangle = \langle \pi(X^n).1, x^m\rangle =$$

$$= \langle 1, \pi(X^n)^*.x^m\rangle = \langle 1, \pi(\partial^n)\, x^m\rangle$$

$$= \langle 1, \underbrace{\left(\frac{d}{dx}\right)^n x^m}_{=0 \text{ since } n>m}\rangle = 0$$

It remains to calculate $\langle x^n, x^n\rangle$:

$$\langle x^n, x^n\rangle = \langle \pi(X^n).1, x^n\rangle =$$

$$= \langle 1, \pi(X^n)^* x^n\rangle = \langle 1, \pi(\partial^n) x^n\rangle =$$

$$= \langle 1, \left(\frac{d}{dx}\right)^n x^n\rangle = n!\langle 1, 1\rangle = n!$$

So $\boxed{\langle x^m, x^n\rangle = n!\,\delta_{m,n}}$

Conversely, one checks that this form satisfies condition (1) on previous page.

Note $\left\{\frac{1}{\sqrt{n!}}x^n\right\}_{n=0}^{\infty}$ is an ON-basis.