LIE GROUPS AND STRATIFIED SPACES

BEN SHELLER

Let G be a Lie group acting on a smooth manifold M. That is, there exists a smooth map $\Phi : G \times M \to M; (g, m) \mapsto g \cdot m$ such that $g \cdot (h \cdot m) = (gh) \cdot m$. An action is called *proper* if for each compact subset $A \subseteq M$, $\Phi^{-1}(A)$ is compact. A nice fact in the theory of Lie group actions is that if an action is both proper and free (meaning that it has only trivial stabilizers), then the quotient M/G is again a manifold. However, if the actions is only assumed to be proper, then the quotient need not be a manifold. For example, if we think of the Lie group $SU(2)$ as a three-dimensional real manifold, with the subgroup $\mathfrak{D}$ of diagonal matrices acting via conjugation, then this action is proper (because $\mathfrak{D}$ is topologically the same as the unit circle, hence compact), and not free (for example, the matrix diagonal$(-1, -1)$ fixes everything), and the result of quotienting out by the action is the unit disc: no longer a smooth manifold! Instead, what is obtained is a *stratified space*. Loosely speaking, a stratified space is very similar to a manifold-with-boundary, except that the boundary may have non-empty boundary as well (and turtles on down), and boundaries need not be of codimension one, as happens in manifolds-with-boundary. We will give a more precise definition a little later, but note that in the example given, this is exactly the case (the unit disc has a boundary of codimension 1). Furthermore, the stratified structure of a quotient by a proper action gives information on the geometry of the original manifold. For example, if the action is both proper and free, then M is a left principal fibre bundle with structure group G (over the quotient manifold M/G). It turns out that in general, if you quotient out a manifold by a proper Lie group action, then the strata of the quotient M/G correspond to "orbits of the family of all vector fields on M/G."

First, let us define the notion of a *differential space*. A differential space $(S, C^\infty S)$ is a topological space S together with a family $C^\infty S$ of real-valued functions on S satisfying:

1. $\{f^{-1}(I) | f \in C^\infty S$ and $I \subseteq \mathbb{R}$ is an open interval$\}$ is a subbasis for the topology on S.
2. If $f_1, ..., f_n \in C^\infty S$ and $F \in C^\infty \mathbb{R}^n$ then $F(f_1, ..., f_n) \in C^\infty S$.
3. If $f : S \to \mathbb{R}$ is a function such that for each $x \in S$, there exists an open neighborhood U of x and a function $f_x \in C^\infty S$ satisfying $f_x|_U = f|_U$ then $f \in C^\infty S$.

For example, any manifold M is a differential space $(M, C^\infty M)$ where $C^\infty M$ denotes the usual collection of smooth functions on M. A differential space is called subcartesian if it is Hausdorff and every $x \in S$ has a neighborhood U

Date: February 24, 2017.

diffeomorphic to a subset V of $\mathbb{R}^n$. Note that V may not be open.
One of the reasons that manifolds are such nice structures is that they come equipped with a tangent bundle, allowing one to do many linear-algebra type arguments. The corresponding notion for a subcartesian space $(S, C^\infty S)$ (or differential space in general) is that of the set of derivations of $C^\infty S$. A derivation at a point $x \in S$ is a linear map $X_x : C^\infty S \to \mathbb{R}$ such that for every $f, h \in C^\infty S$, $X_x(fh) = X_x(f)h(x) + f(x)X_x(h)$. A derivation of $C^\infty S$ is then a collection X of derivations, one at each point of $x \in S$ (so $X : S \times C^\infty S \to \mathbb{R}; (x, f) \mapsto X_x(f)$). We denote the set of all derivations on $C^\infty S$ as $\mathrm{Der}(C^\infty S)$. An integral curve of a derivation $X \in C^\infty S$ is a smooth map $c : I \to S$ from a non-empty interval $I \subseteq \mathbb{R}$ if $\frac{d}{dt} f(c(t)) = X_{c(t)}(f)$ for every $f \in C^\infty S$ and $t \in I$. For each $x \in S$ and $X \in \mathrm{Der}(C^\infty S)$ there exists a unique maximal integral curve c of X such that $0 \in I$ and $c(0) = x$. For c the maximal integral curve of X at x, we denote $c(t)$ by $\exp(tX)(x)$. A vector field on a subcartesian space $(S, C^\infty S)$ is an element $X \in \mathrm{Der}(C^\infty S)$ such that there exist an open neighborhood U of x and $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$, $\exp(tX)$ is defined on U and $\exp(tX)|_U$ is a diffeomorphism from U onto an open subset of S. An *orbit* of a family $\mathfrak{F}$ of vector fields on a subcartesian space S is defined as follows: Let $X_1, ..., X_n \in \mathfrak{F}$ and $x_0 \in S$. Define a piecewise smooth curve given by first following the integral curve of X_1 through x_0 for time t_1, then following the integral curve of X_2 through $x_1 = \exp(t_1 X_1)(x_0)$ for time t_2, then X_3 through $x_2 = \exp(t_2 X_2)(x_1)$ for a time t_3, and so on. For $j = 1, ..., n$, let I_j be the closed interval in $\mathbb{R}$ with endpoints 0 and t_j. Then the orbit of $\mathfrak{F}$ through x_0 is defined as:

$$O_{x_0} = \bigcup_{n=1}^{\infty} \bigcup_{X_1,...,X_n \in \mathfrak{F}} \bigcup_{I_1,...,I_n} \bigcup_{j=1}^{n} \{\exp(t_j X_j)(x_{j-1}) \in S | t_j \in I_j\}$$

where $x_j = \exp(t_j X_j)(x_{j-1})$. It will turn out that the stratified structure on M/G gives information on the orbits of vector fields on M/G.
A stratified space $(S, \mathfrak{L})$ is a second-countable subcartesian differential space $(S, C^\infty S)$ with a collection $\mathfrak{L}$ of smooth manifolds with $\bigcup_{M \in \mathfrak{L}} M = S$ subject to:

1. $\mathfrak{L}$ is **locally closed**: for each $M \in \mathfrak{L}$ and $x \in M$, there exists a neighborhood U of x in S such that $M \cap U$ is closed in U.
2. $\mathfrak{L}$ is **locally finite**: for each $x \in S$ there exists a neighborhood U of x in S such that U intersects only a finite number of manifolds $M \in \mathfrak{L}$.
3. **The Frontier Condition:** For $M, N \in \mathfrak{L}$, if $M \cap \bar{N} \neq \emptyset$, then either $M = N$ or $M \subseteq \bar{N} \setminus N$, where $\bar{N}$ denotes the closure of N in S.

Note that $C^\infty S \subseteq \bigcup_{M \in \mathfrak{L}} C^\infty M$, but the reverse inclusion need not hold (example: manifolds with boundary).
The basic story is this: given a proper Lie group action $\Phi : G \times M \to M$, M/G is a stratified space with *orbit − type* stratification. The orbit-type stratificaton is constructed as follows: Let H be a compact subgroup of G, and define $M_{(H)} := \{x \in M | G_x = gHg^{-1} \text{ for some } g \in G\}$, where $G_x = \{g \in G | gx = x\}$ is the isotropy group of the point $x \in M$. $M_{(H)}$ is called the subset of M of orbit type H. Let $\mathfrak{M}$ be the family of connected components of $M_{(H)}$ as H varies over all

compact subgroups of G. $\mathfrak{M}$ then yields a stratification of M (not a trivial fact - actually takes quite a bit of work). To obtain a stratification $\mathfrak{L}$ on M/G, we project the stratification $\mathfrak{M}$ from M to M/G.
Sketch of proof that M is a stratified space with the stratification

$$\mathfrak{M} = \{\text{connected components of } M_{(H)} | H \text{ a compact subgroup of } G\}$$

The basic idea behind the proof is to first show that for a given compact subgroup H of G, $M_{(H)}$ is a *local submanifold* of M (meaning that each connected component of $M_{(H)}$ is a submanifold of M). This will be enough to show that $\mathfrak{M}$ actually is a set of manifolds covering[1] M, and that it is locally closed (submanifolds of a manifold are locally closed). The way to show this is to first decompose the Lie algebra of G as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie subalgebra corresponding to H in G and $\mathfrak{m}$ is an orthogonal subspace to it. Using this decomposition, one then shows that there is a diffeomorphism φ from an open neighborhood (W, S_p) in $\mathfrak{m} \times S_p$ onto an open neighborhood U of $p \in \{x \in M | G_x = H\}$, where S_p is a *slice*[2] of the action at p Then one shows that the set S_p^H of H-invariant points of S_p is a local submanifold of S_p, so $W \times S_p^H$ is a local submanifold of $W \times S_p$ and $\varphi^{-1}(W \times S_p^H) = U \cap M_{(H)}$.
Then, to show that $\mathfrak{M}$ is locally finite, one proceeds by induction: if $\dim M = 0$, then M is discrete, since G is assumed to be connected and the action (at least) continuous, every point of M is a fixed point of G, and there is only one orbit type: namely, $M_{(G)} = M$, so $\mathfrak{M}$ is locally finite. Then assuming that we know $\mathfrak{M}$ is locally finite for every $\dim N < m$ for some fixed $m \geq 1$, around each $p \in M$, we construct a G_p-invariant ball in T^*S_p, and this has dimension strictly less than m. Then one shows that by projecting down to M, using the local finiteness of the inductive assumption, and gluing the balls together, each point in M is only contained in finitely many components of $M_{(H)}$ for finitely many compact subgroups H of G.
Finally, to show that $\mathfrak{M}$ satisifies the frontier condition requires quite a bit of technical details in looking at a horizontal distribution related to the action of G, which arises in the construction of S_p using a G-invariant Riemannian metric. See [2], section 4.2.
It follows (after a little work showing that the quotient of these orbit-type strata is still a manifold) that M/G is a stratified space with stratification $\pi(\mathfrak{M})$.
Example [1]**:** In the case of $\mathfrak{D}$ acting on $SU(2)$ as above, $SU(2)_{(H)} \neq \emptyset$ if and only if $H = \mathfrak{D}$ or $H = \{I, -I\}$. This yields the stratification $\mathfrak{M} = \{$diagonal matrices, matrices with non-zero off-diagonal elements$\}$. Taking the quotient

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[1]We know that it covers M – follows directly from the definition of a proper action that G_p is compact for each $p \in M$.

[2]Meaning: S_p is a submanifold of M containing p such that:

1. $T_pM = T_pS_p \oplus T_p(Gp)$, where $Gp = \{g \cdot p | g \in G\}$.
2. For every $q \in S_p$, $T_qM = T_qS_p + T_q(Gq)$.
3. S_p is G_p invariant.
4. For $q \in S_p$ and $g \in G$, if $g \cdot q \in S_p$, then $g \in G_p$.

Such a slice can be constructed from a G-invariant Riemannian metric (deliberately vague).

$SU(2)/\mathfrak{D}$ then yields the stratification:

$$\{S_2 := \{[A]_{\alpha,b} : |\alpha|^2 + b = 1, b \in (0,1]\}, S_1 := \{[A]_{\alpha,0} : |\alpha|^2 = 1\}\}$$

where $[A]_{\alpha,b}$ is a parametrization of the equivalence classes in $SU(2)/\mathfrak{D}$ with $\alpha \in \overline{D(0,1)} \subset \mathbb{C}$ and $b \in [0,1]$, where the off-diagonal elements have norm b, and the diagonal is diagonal$(\alpha, \bar{\alpha})$. Note that as real manifolds S_2 has dimension 2, S_1 has dimension 1, and S_1 is the boundary of S_2. In fact, it is easy to see that you can smoothly map S_2 to the interior of the unit disc and S_1 to its boundary. (Just map α!)

Theorem: Strata of the orbit type stratification of M/G are orbits of the family of all vector fields on M/G.

References

[1] Albertini, F. and D'Alessandro, D., *On symmetries in time optimal control, sub-Riemannian geometries and the $K-P$ problem*
[2] Śniatycki J., *Differential Geometry of Singular Spaces and Reduction of Symmetry*

E-mail address: bsheller@iastate.edu

Department of Mathematics, Iowa State University, Ames, 50011, USA