

Gelfand-Tsetlin Bases

- ① Finite-dimensional simple \mathfrak{gl}_n -modules.
- ② $\mathbb{Z}(U(\mathfrak{gl}_n))$ and the Harish-Chandra homomorphism.
- ③ Branching rules $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$
- ④ Rational matrix coefficients.

$$\textcircled{1} \quad \mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{gl}_n = \mathfrak{g}$$

Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ the induced module (Verma module)

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} \mathbb{1}_\lambda$$

$$\text{where } \begin{cases} E_{ii} \mathbb{1}_\lambda = \lambda_i \mathbb{1}_\lambda \\ E_{ij} \mathbb{1}_\lambda = 0 \quad i < j \end{cases}$$

has a unique simple quotient denoted $V(\lambda)$.

Thm.

(i) $\dim V(\lambda) < \infty$ iff $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, 2, \dots, n-1$

(ii) Any finite-dimensional simple \mathfrak{gl}_n -module is isomorphic to $V(\lambda)$, for some λ .

(iii) $V(\lambda) \cong V(\lambda') \Leftrightarrow \lambda = \lambda'$

② Put for $k = (k_{ij})_{1 \leq i < j \leq n}, k_{ij} \in \mathbb{Z}_{\geq 0}$

$$E^k = E_{12}^{k_{12}} E_{13}^{k_{13}} \dots E_{1n}^{k_{1n}} \cdot E_{23}^{k_{23}} \dots E_{2n}^{k_{2n}} \dots \cdot E_{n-1,n}^{k_{n-1,n}}$$

Similarly we put for $l = (l_{ij})_{1 \leq j < i \leq n}$

$$F^l = E_{21}^{l_{21}} \dots E_{n,n-1}^{l_{n,n-1}}$$

By the PBW theorem

$$U(\mathfrak{gl}_n) = \bigoplus_{k,l,a} \mathbb{C} F^l \cdot E_{11}^{a_1} \dots E_{nn}^{a_n} \cdot E^k$$

$$\cong U(\mathfrak{m}^-) \otimes_{\mathbb{C}} U(\mathfrak{h}) \otimes U(\mathfrak{m}^+) \quad \text{as v.s.p.}$$

Lemma $(\mathfrak{m}^- U(\mathfrak{g})) \cap Z(U(\mathfrak{gl}_n)) =$
 $= (U(\mathfrak{g}) \mathfrak{m}^+) \cap Z(U(\mathfrak{gl}_n))$

Proof Let $x \in \text{LHS}$.

$$x = \sum_{k,l,a} \underbrace{x_{k,l,a}}_{\in \mathbb{C}} F^l E_{11}^{a_1} \dots E_{nn}^{a_n} E^k$$

where $x_{k,0,a} = 0 \quad \forall k,a$

$$0 = [E_{ii}, x] \Rightarrow x_{0,l,a} = 0 \quad \forall l,a$$

\uparrow
 $x \in Z(U(\mathfrak{gl}_n))$

Q.E.D.

Thm (Harish-Chandra)

The restriction of the projection

$$\varphi: U(\mathfrak{gl}_n) \rightarrow U(\mathfrak{h})$$

$$F^l E_{11}^{a_1} \dots E_{nn}^{a_n} E^k \mapsto \begin{cases} E_{11}^{a_1} \dots E_{nn}^{a_n}, & l=k=0 \\ 0, & \text{otherwise} \end{cases}$$

to $Z(U(\mathfrak{gl}_n))$ is an algebra

homomorphism. Moreover: $Z(U(\mathfrak{gl}_n)) \xrightarrow{\cong} U(\mathfrak{h})^{S_n}$

Pf $Z = Z_0 + Z_1$
 $k=l=0$
 $Z_0 = \varphi(z)$
 $Z_1 = z - \varphi(z)$
 $W = W_0 + W_1$
 $\delta(E_{ii} - i) = E_{\sigma(i)\sigma(i)}^{-\sigma(i)}$

$$z w = z_0 w_0 + z_1 w_0 + z_0 z w_1 + (z_1 w_1)$$

$$\begin{array}{ccc} \downarrow & \downarrow \varphi & \downarrow \\ z_0 w_0 & 0 & 0 \end{array}$$

$$\in \mathfrak{h}^- U(\mathfrak{g}) \cdot U(\mathfrak{g}) \mathfrak{h}^+ \subseteq \mathfrak{h}^- U(\mathfrak{g})$$

hence $\varphi(z_1 w_1) = 0$

So $\varphi(zw) = \varphi(z)\varphi(w)$.

We skip the second part.

QED

Put $Z(\mathfrak{g}) = \mathbb{C}[U(\mathfrak{g})]$

Corollary If $z \in Z(\mathfrak{g})$ then in $V(\lambda)$:

$$z \cdot \mathbb{1}_\lambda = \varphi(z) \cdot \mathbb{1}_\lambda$$

Corollary There exists $d_{ni} \in Z(\mathfrak{g})$

$Z(\mathfrak{g}) \cong \mathbb{C}[d_{ni} \mid 1 \leq i \leq n]$ pol. alg &

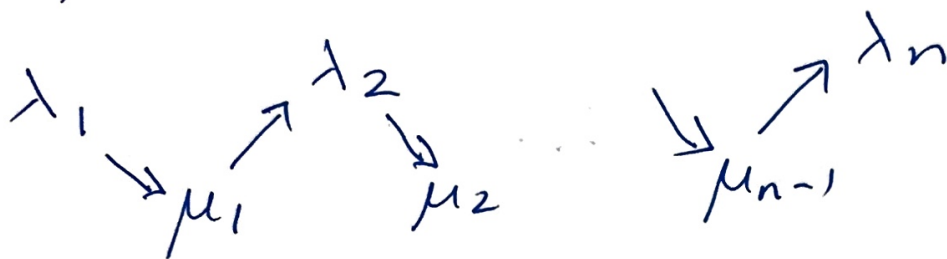
$$d_{ni} \cdot \mathbb{1}_\lambda = e_{ni}(\lambda_1 - 1, \dots, \lambda_{n-n}) \mathbb{1}_\lambda$$

↑ el. sym. pol of deg i

③ As \mathfrak{gl}_{n-1} modules

$$V(\lambda) \cong \bigoplus_{\substack{\mu \in \mathbb{C}^{n-1} \\ \lambda \downarrow \mu}} V(\mu)$$

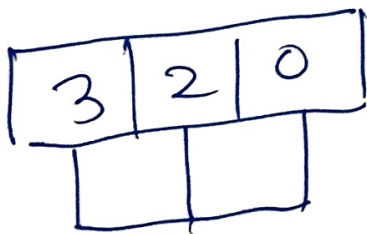
$\lambda \downarrow \mu$ means



$a \rightarrow b$
means
 $a - b \in \mathbb{Z}_{\geq 0}$

Ex. $\lambda = (3, 2, 0)$ Then

$$V(\lambda) \cong V(3, 2) \oplus V(3, 1) \oplus V(3, 0) \\ \oplus V(2, 2) \oplus V(2, 1) \oplus V(2, 0)$$



④

Repeating this process we obtain:

Any fin. dim \mathfrak{gl}_n -module is a direct sum of 1-dimensional subspaces parametrized by Gelfand-Tsetlin patterns:

$$V(3, 2, 0) = \begin{array}{|c|c|c|} \hline 3 & 2 & 0 \\ \hline 3 & 2 & \\ \hline 3 & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline 3 & 2 & 0 \\ \hline 3 & 2 & \\ \hline 2 & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline 3 & 2 & 0 \\ \hline 3 & 1 & \\ \hline 3 & & \\ \hline \end{array} \oplus \dots$$

Thm (Gelfand-Tsetlin 1950)

There is a choice of basis $\{v_\lambda \mid \lambda \text{ pattern}\}$ for $V(\lambda)$ in which all matrix coefficients $\langle E_{ij} v_\lambda, v_{\lambda'} \rangle$ are rational functions of t_{ij} .