# Gelfand-Tsetlin Bases

1. Finite-dimensional simple $\mathfrak{gl}_n$-modules.

2. $Z(U(\mathfrak{gl}_n))$ and the Harish-Chandra homomorphism.

3. Branching rules $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$

4. Rational matrix coefficients.

① $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{gl}_n = \mathfrak{g}$

Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ the induced module (verma module)

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\mathbb{1}_\lambda$$

where $\begin{cases} E_{ii}\mathbb{1}_\lambda = \lambda_i \mathbb{1}_\lambda \\ E_{ij}\mathbb{1}_\lambda = 0 & i<j \end{cases}$

has a unique simple quotient denoted $V(\lambda)$.

<u>Thm</u>.

(i) $\dim V(\lambda) < \infty$ iff $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, 2, \dots, n-1$

(ii) Any finite-dimensional simple $\mathfrak{gl}_n$-module is isomorphic to $V(\lambda)$, for some $\lambda$.

(iii) $V(\lambda) \cong V(\lambda') \iff \lambda = \lambda'$

(2) Put for $k = (k_{ij})_{1 \le i < j \le n}$, $k_{ij} \in \mathbb{Z}_{\ge 0}$

$$E^k = E_{12}^{k_{12}} E_{13}^{k_{13}} \cdots E_{1n}^{k_{1n}} \cdot E_{23}^{k_{23}} \cdots E_{2n}^{k_{2n}} \cdot \dots \cdot E_{n-1,n}^{k_{n-1,n}}$$

Similarly we put for $l = (l_{ij})_{1 \le j < i \le n}$

$$F^l = E_{21}^{l_{21}} \cdots E_{n,n-1}^{l_{n,n-1}}$$

By the PBW theorem

$$U(\mathfrak{gl}_n) = \bigoplus_{k,l,a} \mathbb{C}\, F^l \cdot E_{11}^{a_1} \cdots E_{nn}^{a_n} \cdot E^k$$

$$\cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \quad \text{as v.sp.}$$

<u>Lemma</u> $(\mathfrak{n}^- U(\mathfrak{g})) \cap Z(U(\mathfrak{gl}_n)) =$
$= (U(\mathfrak{g})\mathfrak{n}^+) \cap Z(U(\mathfrak{gl}_n))$

<u>Proof</u> Let $x \in$ LHS.

$$x = \sum_{k,l,a} \underbrace{x_{k,l,a}}_{\in \mathbb{C}} F^l E_{11}^{a_1} \cdots E_{nn}^{a_n} E^k$$

where $x_{k,0,a} = 0 \quad \forall k, a$

$$\underset{\substack{\uparrow \\ x \in Z(U(\mathfrak{gl}_n))}}{0 =} [E_{ii}, x] \Rightarrow x_{0,l,a} = 0 \quad \forall l, a$$

Q.E.D.

**Thm** (Harish-Chandra)

The restriction of the projection

$$\varphi : U(\mathfrak{gl}_n) \longrightarrow U(\mathfrak{h})$$

$$F^{l} E_{11}^{a_1} \cdots E_{nn}^{a_n} E^{k} \longmapsto \begin{cases} E_{11}^{a_1} \cdots E_{nn}^{a_n}, & l=k=0 \\ 0, & \text{otherwise} \end{cases}$$

to $Z(U(\mathfrak{gl}_n))$ is an algebra homomorphism. **Moreover**: $Z(U(\mathfrak{gl}_n)) \xrightarrow[\varphi]{\cong} U(\mathfrak{h})^{\cdot S_n}$

$\sigma \cdot (E_{ii} - i) = E_{\sigma(i)\sigma(i)} - \sigma(i)$

Pf $\quad z = z_0 + z_1$, $\quad w = w_0 + w_1$

$k = l = 0$

$z_0 = \varphi(z)$
$z_1 = z - \varphi(z)$

$$zw = z_0 w_0 + z_1 w_0 + z_0 w_1 + z_1 w_1$$

$z_0 w_0 \overset{\varphi}{\mapsto} z_0 w_0$, $\quad z_1 w_0 \overset{\varphi}{\mapsto} 0$, $\quad z_0 w_1 \overset{\varphi}{\mapsto} 0$

$z_1 w_1 \in \mathfrak{n}^- U(\mathfrak{g}) \cdot U(\mathfrak{g}) \mathfrak{n}^+ \subseteq \mathfrak{n}^- U(\mathfrak{g})$

hence $\varphi(z_1 w_1) = 0$

So $\varphi(zw) = \varphi(z)\varphi(w)$.

We skip the second part. QED

Put $Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

Corollary If $z \in Z(\mathfrak{g})$ then in $V(\lambda)$:

$$z \cdot \mathbb{1}_\lambda = \varphi(z) \cdot \mathbb{1}_\lambda$$

Corollary There exists $d_{ni} \in Z(\mathfrak{g})$

$Z(\mathfrak{g}) \cong \mathbb{C}[d_{ni} \mid 1 \leq i \leq n]$ pol. alg &

$$d_{ni} \cdot \mathbb{1}_\lambda = e_{ni}(\lambda_1 - 1, \dots, \lambda_n - n) \mathbb{1}_\lambda$$

el. sym. pol of deg $i$

(3) As $\mathfrak{gl}_{n-1}$ modules

$$V(\lambda) \cong \bigoplus_{\substack{\mu \in \mathbb{C}^{n-1} \\ \lambda \Downarrow \mu}} V(\mu)$$

$\lambda \Downarrow \mu$ means

$$\lambda_1 \to \mu_1 \to \lambda_2 \to \mu_2 \quad \cdots \quad \to \mu_{n-1} \to \lambda_n$$

$a \to b$ means $a - b \in \mathbb{Z}_{\geq 0}$

Ex. $\lambda = (3, 2, 0)$ Then

$$V(\lambda) \cong V(3,2) \oplus V(3,1) \oplus V(3,0) \oplus V(2,2) \oplus (2,1) \oplus V(2,0)$$

| | | |
|---|---|---|
| 3 | 2 | 0 |
| | | |

Repeating this process we obtain:

Any fin. dim $\mathfrak{gl}_n$-module is a direct sum of 1-dimensional subspaces parametrized by <u>Gelfand-Tsetlin patterns</u>:

$$V(3,2,0) = \begin{array}{c} \boxed{3\,|\,2\,|\,0} \\ \boxed{3\,|\,2} \\ \boxed{3} \end{array} \oplus \begin{array}{c} \boxed{3\,|\,2\,|\,0} \\ \boxed{3\,|\,2} \\ \boxed{2} \end{array} \oplus$$

$$\oplus \begin{array}{c} \boxed{3\,|\,2\,|\,0} \\ \boxed{3\,|\,1} \\ \boxed{3} \end{array} \oplus \cdots\cdots$$

<u>Thm</u> (Gelfand-Tsetlin 1950)
There is a choice of basis $\{v_\Lambda \mid \Lambda \text{ pattern}\}$ for $V(\lambda)$ in which all matrix coefficients $\langle E_{ij} v_\Lambda, v_{\Lambda'} \rangle$ are rational functions of $\lambda_{ij}$.