LIE THEORY SEMINAR 11/4/2016

What about E_9? Kac-Moody algebras

Cartan matrices

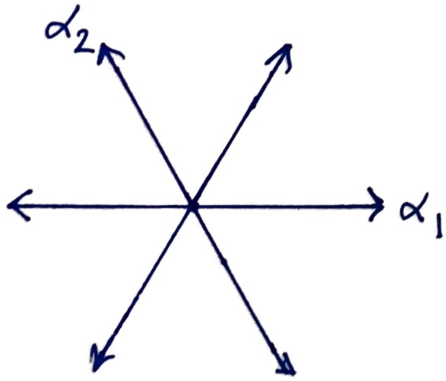
Let Φ be a root system and $\{\alpha_1, \dots, \alpha_\ell\}$ a set of simple roots.

The corresponding Cartan matrix $A = (a_{ij})_{1 \le i,j \le \ell}$ is given by

$$a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

where $(\cdot, \cdot)$ is the inner product.

Ex



$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Note the following connection to sl_3:

$$[h_i, e_j] = a_{ji} e_j \qquad h_1 = \begin{matrix} 1 & & \\ & -1 & \\ & & 0 \end{matrix} \quad h_2 = \begin{matrix} 0 & & \\ & 1 & \\ & & -1 \end{matrix}$$

$$e_1 = \begin{matrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{matrix} \quad e_2 = \begin{matrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{matrix}$$

Properties of $A = (a_{ij})_{1 \le i,j \le \ell}$

i) $a_{ii} = 2 \quad \forall i$

ii) $a_{ij} \in \mathbb{Z}_{\le 0}$ if $i \ne j$

iii) $a_{ij} = 0 \iff a_{ji} = 0$ (symmetric zeros)

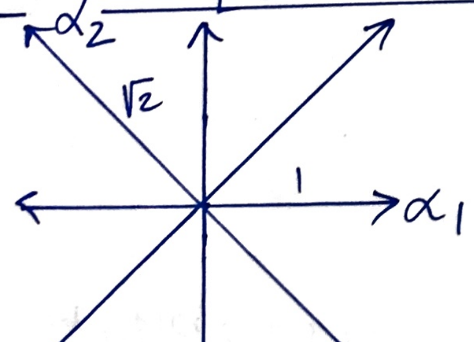
iv) A is positive definite.

($\iff$ all principal minors are > 0)

Thm Any matrix satisfying i) – iv) comes from a unique root system. Moreover, A is indecomposable $\iff \Phi$ irreducible.

(not $\sim \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$)

EX



B_2

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$a_{12} = \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{\cancel{2} \cdot \overbrace{\|\alpha_1\|}^{1} \cdot \overbrace{\|\alpha_2\|}^{\sqrt{2}} \cos\theta}{\cancel{\|\alpha_2\|^2}} = \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -1$$

$$a_{21} = \frac{2(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{2\|\alpha_2\| \cdot \|\alpha_1\| \cos\theta}{\|\alpha_1\|^2} = \frac{2\sqrt{2} \cdot 1 \cdot \left(-\frac{1}{\sqrt{2}}\right)}{1^2} = -2$$

<u>Thm</u> (Chevalley-Serre)

Let $\mathfrak{g}$ be a f.d. simple Lie alg $/\mathbb{C}$, and C its cartan matrix.

Then $\mathfrak{g}$ is generated by

$$\{e_1, \dots, e_\ell\} \cup \{f_1, \dots, f_\ell\} \cup \{h_1, \dots, h_\ell\}$$

subject to the relations

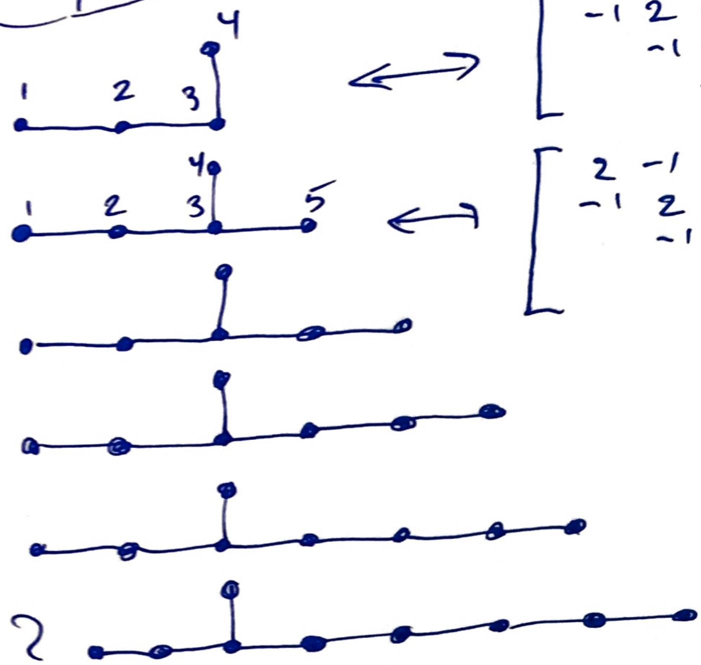
$$\left.\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= a_{ji} e_j \\ [h_i, f_j] &= -a_{ji} f_j \end{aligned}\right\} \text{Chevalley Relations}$$

$$\left.\begin{aligned} (\operatorname{ad} e_i)^{1-a_{ji}}(e_j) &= 0 \qquad i \neq j \\ (\operatorname{ad} f_i)^{1-a_{ji}}(f_j) &= 0 \qquad i \neq j \end{aligned}\right\} \text{Serre relations}$$

<u>Note</u> $(\operatorname{ad} e_i)^{1-a_{ji}}(e_j) = [e_i [e_i \cdots [e_i, e_j]] \cdots]$

$$\begin{tikzcd} \text{Fin.dim simple Lie algs} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \text{Irreducible Root Systems} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Connected Dynkin Diagrams} \\ & \text{Indecomposable Cartan matrices} & \end{tikzcd}$$

E_9 ? Dynkin: Cartan: Determinant



	Cartan	Determinant
A_4	$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$	5
D_5	$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & -1 \\ & & -1 & 2 & \\ & & -1 & & 2 \end{bmatrix}$	4
E_6		3
E_7		2
E_8		1
E_9 ?		0

So the "cartan matrix" of E_9 is not positive definite.

<u>Question</u>: What if we <u>drop</u> (iv) in the definition of cartan matrix? What do we get if we define a Lie alg through Chevalley-Serre presentation starting from such a matrix?

$\boxed{\text{Kac-Moody algebras}}$

Def A <u>generalized Cartan matrix</u> (GCM) is a matrix $A = (a_{ij})_{1 \le i,j \le n}$ satisfying i) – iii):

i) 2's on diagonal

ii) nonpos integer off-diagonal

iii) symmetric zeros

Def Let A be an $n \times n$ GCM of rank ℓ. The <u>Kac-Moody alg</u> $\mathfrak{g}(A)$ is the Lie alg with generators

$h_1, h_2, \ldots, h_{2n-\ell}$

$e_1, \ldots, e_n$

$f_1, \ldots, f_n$

subject to relations

<u>Extended GCM</u>:

nonsingular

$$\hat{A} = \begin{array}{cc} & \begin{array}{ccc} \ell & n-\ell & \end{array} \\ \ell & \left[\begin{array}{cc|c} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ \hline 0 & I_{n-\ell} & 0 \end{array}\right] \end{array}$$

$$\begin{cases} [h_i, h_j] = 0, & [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] = a_{ji} e_j & [h_i, f_j] = -a_{ji} f_j \\ (\operatorname{ad} e_i)^{1-a_{ji}}(e_j) = 0 & (\operatorname{ad} f_i)^{1-a_{ji}}(f_j) = 0 \end{cases}$$

where a_{ji} are entries of $\hat{A}$.

Def $A=(a_{ij})$ is <u>affine type</u> if $\operatorname{rank} A = n-1$ & $\exists\, u_1>0,\dots,u_n>0$ with $A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = 0$.

EX GCM of Eq.

Thm (Loop alg realization) A affine. Then

$$\mathfrak{g}(A) \cong \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$[x\otimes t^m, y\otimes t^n] = [x,y]\otimes t^{m+n} + \delta_{m+n,0}(x,y)c$$

$$[c, \mathfrak{g}(A)] = 0$$

$$[d, x\otimes t^m] = m\, x\otimes t^m \quad \forall x,y \in \mathfrak{g}.$$

(or twisted subalg)

These so called <u>affine Kac-Moody algebras</u> are infinite-dimensional Lie algebras with applications in number theory and conformal field theory (VOAs).