Lie superalgebras and superdifferential operators (Part 1/2)

"Super" = $\mathbb{Z}_2$-graded, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

I. Motivation

– Physics: Fermions = matter, ex. electrons, $xy = -yx$
Bosons = force carriers, ex. photons, $xy = yx$
Pauli Exclusion Principle: Identical fermions can't occupy the same quantum state
$\psi(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = (\operatorname{sgn}\sigma)\psi(x_1, x_2, \dots, x_n)$

Ex: If x_1, x_2, y fermionic then
$(x_1 x_2)y = -x_1 y x_2 = y(x_1 x_2)$
i.e. pairs of fermions behave like a boson (cooper pairs $\Rightarrow$ superconductivity)
Also important in supersymmetric (SUSY) string theory.

– Mathematics: 1) V vector space $\Lambda V = \bigoplus_{k=0}^{\dim V} \Lambda^k V$ the exterior algebra is $\mathbb{Z}_2$-graded:

$$\Lambda V = \left(\bigoplus_{k=0}^{\infty} \Lambda^{2k} V\right) \oplus \left(\bigoplus_{k=0}^{\infty} \Lambda^{2k+1} V\right)$$

2) The cohomology ring $H^*(X)$ of any topological space X is graded (= super) commutative.

II. Superalgebra.

Def A vector superspace is a vector space with a distinguished decomposition

$$V = \underset{\text{even}}{V_{\bar{0}}} \oplus \underset{\text{odd}}{V_{\bar{1}}}$$

Def A linear map $L : V \to V$ is

even if $L(V_{\bar{a}}) \subseteq V_{\bar{a}}$

odd if $L(V_{\bar{a}}) \subseteq V_{\bar{a}+\bar{1}}$

In matrix form relative to a basis

$$\underbrace{\{v_1, \dots, v_m\}}_{V_{\bar{0}}} \cup \underbrace{\{v_{m+1}, \dots, v_{m+n}\}}_{V_{\bar{1}}}$$

$$L \text{ even} : \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$$

$$L \text{ odd} : \left[\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right]$$

Thus $\mathrm{End}(V) = \underset{\text{even } L}{\mathrm{End}(V)_{\bar{0}}} \oplus \underset{\text{odd } L}{\mathrm{End}(V)_{\bar{1}}}$

$\Rightarrow \mathrm{End}(V)$ also a vector superspace!

Note If $a \in End(V)_\alpha$, $b \in End(V)_\beta$ then $ab \in End(V)_{\alpha+\beta}$

Definition An (associative) superalgebra is a vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with an (associative) bilinear operation $A \times A \to A$ such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$

Sign Rule When generalizing classical notions to their super analogs,

(i) Write formulas only for homogeneous elements (i.e. $\in A_\alpha \cup A_\beta$) & extend linearly

(ii) whenever $a \in A_\alpha$ $b \in A_\beta$ are switched, $(-1)^{\alpha\beta}$ appears.

Ex:

1) Def A superalgebra A is commutative if $ab = (-1)^{\alpha\beta} ba$ $\forall a \in A_\alpha, b \in A_\beta$.

Ex ΛV is a commutative superalg.

2) A linear operator $D \in \mathrm{End}(A)_\delta$ is a <u>superderivation</u> if

$$D(ab) = D(a)b + (-1)^{\alpha\delta} a D(b) \qquad \forall a \in A_\alpha, \; b \in A_\beta$$

$\uparrow$

a and D have "switched places"

3) Let V be a vector superspace. The <u>symmetric superalgebra</u> $S(V)$ is

$$T(V) \Big/ \langle a \otimes b - (-1)^{\alpha\beta} b \otimes a \mid a \in A_\alpha \; b \in A_\beta \rangle$$

<u>Note</u> If $V = V_{\bar{1}}$ (purely odd) then $S(V)$ is really the exterior algebra.

4) An <u>even symmetric</u> bilinear form $(\cdot, \cdot)$ on $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a bil. form satisfying

$$(a, b) = (-1)^{\alpha\beta}(b, a) \qquad a \in V_\alpha, \; b \in V_\beta$$

$$(V_{\bar{0}}, V_{\bar{1}}) = 0$$

III. Lie superalgebras

Def A Lie superalg is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t.

i) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

ii) $[a, b] = -(-1)^{\alpha\beta} [b, a]$

iii) $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$

Note: iii) says that the map

$\mathrm{ad}\, a : \mathfrak{g} \to \mathfrak{g}$, $(\mathrm{ad}\, a)(b) = [a,b]$

is a superderivation on $\mathfrak{g}$.

Ex If $V = V_{\bar{0}} \oplus V_{\bar{1}}$, $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$

then $\mathfrak{gl}(m|n) = \mathrm{End}(V)$ is a Lie superalgebra with $[a,b] = ab - (-1)^{\alpha\beta} ba$.

Ex The supertrace of $X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathrm{End}(V)$

is $\mathrm{str}\, X = \mathrm{tr}\, A - \mathrm{tr}\, D$.

$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) \mid \mathrm{str}\, X = 0\}$

Lie subalg of $\mathfrak{gl}(m|n)$.

Ex. $\mathfrak{osp}(m|2n) = \{x \in \mathfrak{gl}(m|n) \mid$

$(xa, b) + (-1)^{\xi a}(a, xb) = 0$

for all homogeneous a, b in $V\}$

where $(\cdot,\cdot)$ even symm nondeg bil form on $V = V_{\bar 0} \oplus V_{\bar 1}$

$\dim$ m $2n$

Def $\mathfrak{g}$ simple if it has no nontriv proper ideals.

<u>Thm</u> (Kac 1977)

Any simple fd Lie superalg over $\mathbb{C}$ is isomorphic to one of:

I <u>contragredient</u> - Basic classical
$\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|2n)$
- Exceptional ($\mathfrak{g}_{\bar{1}} \neq 0$)
$D(2,1;\alpha)$ $G(1|2)$, $F(1|3)$

II <u>classical strange</u> $\mathfrak{q}(n)$, $P(n)$

III <u>Cartan type</u> $W(0|n)$, $S(n)$, $S'(n)$, $H(n)$