Plan of talk

1. Definition of Lie algebra and homomorphisms

2. Examples

3. Classification problem

[All vector spaces will be over $\mathbb{C}$]

Def A Lie algebra L is a vector space together with a map

$$L \times L \to L \ , \ (x,y) \mapsto [x,y]$$

called the bracket such that

(i) $[\lambda x + \mu y, z] = \lambda [x,z] + \mu [y,z]$

$[x, \lambda y + \mu z] = \lambda [x,y] + \mu [x,z]$

$\forall x,y,z \in L \quad \forall \lambda, \mu \in \mathbb{C}$ (bilinearity)

(ii) $[x,y] = -[y,x] \quad \forall x,y \in L$ (skew-symmetry)

(iii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

$\forall x,y,z \in L$ (Jacobi Identity)

<u>Def</u> A <u>homomorphism</u> $\varphi: L_1 \to L_2$ where L_1 and L_2 are Lie algebras is a linear map such that

$$\varphi([x,y]_{L_1}) = [\varphi(x), \varphi(y)]_{L_2}$$

$\forall x, y \in L_1$.

<u>Examples</u>

1) Any vector space L can be turned into a Lie algebra in a trivial way: Define

$$[x,y] = 0 \quad \forall x, y \in L.$$

Axioms (i)–(iii) are trivial to check.

A Lie algebra in which $[x,y]=0$ $\forall x,y \in L$ is called <u>abelian</u>.

2) Let $L = M_n(\mathbb{C})$ the vector space of all $n \times n$ matrices.
Define

$$[A,B] = AB - BA \quad \text{for } A, B \in M_n(\mathbb{C})$$

Then (i), (ii) are easy to check while (iii) requires a computation to verify (Do it!)
The resulting Lie algebra is denoted $\mathfrak{gl}_n$ (<u>general linear Lie algebra</u>)

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3) If A is any associative algebra, we define

$$[a,b] = ab - ba \quad \forall a, b \in A.$$

Then A becomes a Lie algebra as in Example 2).

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4) Let

$$sl_n = \{A \in gl_n \mid Tr(A) = 0\}$$

Then sl_n is closed under the bracket: Suppose $A, B \in sl_n$. Then

$$Tr([AB]) = Tr(AB - BA) =$$
$$= Tr(AB) - Tr(BA) = 0$$

So $[A,B] \in sl_n$.

Therefore sl_n is a <u>Lie subalgebra</u> of gl_n, the <u>special linear Lie algebra</u>.

$n=2$: $$sl_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{C} \\ a+d=0 \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{C} \right\} =$$

$$= \left\{ a \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{h} + b \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{e} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{f} \;\middle|\; a,b,c \in \mathbb{C} \right\}$$

<u>Def</u> L_1 and L_2 are isomorphic if there is a bijective homomorphism $\varphi : L_1 \to L_2$

<u>Classification Problem</u>: Classify all Lie algebras up to isomorphism.

<u>Levi Decomposition</u> Any finite-dimensional Lie algebra L can be decomposed

$$L = \mathfrak{g} \oplus I \quad \text{(as vector spaces)}$$

where
$\mathfrak{g}$ is a <u>semi-simple</u> Lie algebra
and
I is a <u>solvable</u> Lie algebra
(and $[\mathfrak{g}, I] \subseteq I$ (I is an <u>ideal</u> of L))

<u>Example</u> $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}I$: $A = \left(A - \frac{\operatorname{Tr} A}{n} I\right) + \frac{\operatorname{Tr} A}{n} I$

Classification problem breaks into (for f.d. Lie algs)

1. Classify all finite-dimensional semi-simple Lie algebras (doable in a single semester course on Lie algebras! very nice)

2. Classify all finite-dimensional solvable Lie algebras (hopeless)

$sl_2 = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Using $[xy] = x \cdot y - y \cdot x$ where $\cdot$ is matrix multiplication, we find

$[e, f] = h$

$[h, e] = 2e$

$[h, f] = -2e$

Using these and $[xy] = -[yx]$ and bilinearity we can compute $[x, y]$ for all $x, y \in sl_2$.