510

Tensor Products $\otimes$

The most important notion in all of science!

QM vs. GR.

Conj: $QM \supseteq GR$

space-time is created via entanglement ← defined using $\otimes$

Recall:

1) For any set X we can create a vector space $\mathbb{F}X$ with basis $\{e_x\}_{x \in X}$ indexed by X.

Ex. $\mathbb{F}\{1,2\} \cong \mathbb{F}^2$

$\mathbb{F}\mathbb{N} \cong \mathbb{F}[x]$

2) If V v. sp., $U \leq V$ then we can form

$$V/U = \{v + U \mid v \in V\}$$

w/ op's $(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$

$\lambda(v + U) = (\lambda v) + U$

Also, $v_1 + U = v_2 + U \iff v_1 - v_2 \in U$.

Let V, W be any two vec. sp's. Goal: Define a new vector space $V \otimes W$.

① Form $\mathbb{F}(V \times W)$. This is a vector space with basis

$$\{e_{(v,w)} \mid v \in V, w \in W\}$$

② Let $U \leq \mathbb{F}(V \times W)$ be the span of the set

$$\{e_{(v_1+v_2,w)} - e_{v_1,w} - e_{v_2,w} \mid \begin{smallmatrix} v_1, v_2 \in V \\ w \in W \end{smallmatrix}\} \cup$$

$$\cup \{e_{v,w_1+w_2} - e_{v,w_1} - e_{v,w_2} \mid \begin{smallmatrix} v \in V \\ w_1, w_2 \in W \end{smallmatrix}\} \cup$$

$$\cup \{e_{\lambda v,w} - \lambda e_{v,w} \mid \begin{smallmatrix} \lambda \in \mathbb{F} \\ v \in V, w \in W \end{smallmatrix}\} \cup \{e_{v,\lambda w} - \lambda e_{v,w} \mid \begin{smallmatrix} \lambda \in \mathbb{F} \\ v \in V \\ w \in W \end{smallmatrix}\}.$$

③ Define

$$V \otimes W := \frac{F(V \times W)}{U}$$

This space is the <u>tensor product</u> of V and W.

<u>Notation</u> : For $v \in V$, $w \in W$ we put

$$v \otimes w = e_{(v,w)} + U \in V \otimes W,$$

<u>Properties.</u> For any $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$, we have:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$(\lambda v) \otimes w = \lambda (v \otimes w)$$

$$v \otimes (\lambda w) = \lambda (v \otimes w).$$

<u>Proof</u> $v_1 \otimes w + v_2 \otimes w = (e_{(v_1,w)} + U) + (e_{(v_2,w)} + U)$

$= (e_{(v_1,w)} \cancel{+U} + e_{(v_2,w)} + U)$

$(v_1 + v_2) \otimes w = e_{(v_1+v_2,w)} + U$. These are equal since $e_{(v_1+v_2,w)} - e_{(v_1,w)} - e_{(v_2,w)} \in U$. The rest similar ∎

<u>Thm</u> if V has basis $\mathcal{B} = (b_1, \ldots, b_n)$ and W has basis

$$\mathcal{C} = (c_1, \ldots, c_m)$$

then $V \otimes W$ has basis

$$\begin{aligned}\mathcal{B} \times \mathcal{C} = (&b_1 \otimes c_1, b_1 \otimes c_2, \ldots, b_1 \otimes c_m,\\ &b_2 \otimes c_1, b_2 \otimes c_2, \ldots, b_2 \otimes c_m,\\ &\ldots, b_n \otimes c_1, b_n \otimes c_2, \ldots, b_n \otimes c_m)\end{aligned}$$

<u>proof</u> $\mathcal{B} \times \mathcal{C}$ spans clear.

$\mathcal{B} \times \mathcal{C}$ lin indep: SKIPPED.

<u>Cor</u> $\dim V \otimes W = (\dim V)(\dim W)$

V, W, X v. sp's

Def A *bilinear map*

$$\beta : V \times W \longrightarrow X$$

is a fcn s.t.

$$\beta(v_1 + v_2, w) = \beta(v_1, w) + \beta(v_2, w)$$

$$\beta(v, w_1 + w_2) = \cdots$$

$$\beta(\lambda v, w) = \lambda \beta(v, w) = \beta(v, \lambda w)$$

Not. $Bil(V \times W, X) = \{\beta\}$

Thm There is a bijection

$$Bil(V \times W, X) \cong Hom(V \otimes W, X).$$

Proof Let $\beta : V \times W \to X$ bil. map.
By property of free vector spaces $\exists !$ linear map $\overline{\beta} : \mathbb{F}V \times W \to X$ given by $\overline{\beta}(e_{(v,w)}) = \beta(v, w)$.
β bil $\Rightarrow U \leq Ker\,\overline{\beta}$. Get induced lin. $B : V \otimes W \longrightarrow X$. Conv., given B, define $\beta(v, w) = B(v \otimes w)$ ∎

<u>Cor</u>

If $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ are lin maps, $\exists !$ lin map

$$L_1 \otimes L_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$$

Subject to

$$v_1 \otimes v_2 \mapsto L_1(v_1) \otimes L_2(v_2)$$

<u>proof</u> Define

$$\beta : V_1 \times V_2 \longrightarrow W_1 \otimes W_2$$

by $\beta(v_1, v_2) = L_1(v_1) \otimes L_2(v_2)$.

Check β bil. Thus $\exists !$ bil. $B = L_1 \otimes L_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ as claimed.

<u>Thm</u>

$$[L_1 \otimes L_2]_{\mathcal{B}_1 \times \mathcal{B}_2, \mathcal{C}_1 \times \mathcal{C}_2} =$$

$$= [L_1]_{\mathcal{B}_1, \mathcal{C}_1} \otimes [L_2]_{\mathcal{B}_2, \mathcal{C}_2}$$

Kronecker product

$$[a_{ij}] \otimes [b_{ij}] = \begin{bmatrix} a_{11}[b_{ij}] & \cdots \\ \vdots & \\ & \end{bmatrix}$$

Ex

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \\ \varepsilon & \eta \end{bmatrix}}_{B} = \left[\begin{array}{cc|c} a\cdot\alpha & a\beta & b\alpha \\ a\gamma & a\delta & b\gamma \\ a\varepsilon & a\eta & b\varepsilon \\ \hline & & \\ & & \end{array}\right]$$

Appl.

Transition probability matrices: Rows & cols are $\in (0,1)$, sum to 1

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \otimes \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} =$$