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Invariant Factors I

$T : V \to V$

$(\dim V = n < \infty)$

To T we can associate

invariants = quantities that only depend on T itself (and not eg choice of basis)

① $m_T(x)$ min pol $\in \mathbb{F}[x]$

② $\det(T)$

③ $\operatorname{rank} T$, (nullity T) $\in \{0, 1, .., n\}$

It turns out $\{m_T(x)\}$ can be enlarged to a set of n pols $\{f_1(x), \ldots, f_n(x)\}$

Called invariant factors of T.

They are monic polys in $\mathbb{F}[x]$ such that

1) $f_{k+1}(x) \mid f_k(x) \quad k=1,\dots,n-1$

(often the last couple f_i's equal 1 and are thrown out)

Ex
$$\begin{bmatrix} (x-3)^2(x-4) & & & & \\ & (x-3)(x-4) & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

2) $f_1(x) = m_T(x)$

3) $f_1(x) f_2(x) \cdots f_n(x) = C_T(x) := \det(x \mathrm{Id}_V - T)$
Characteristic pol.

In particular
$$\sum_{i=1}^{n} \deg f_i = n$$

Aside: Any field $\mathbb{F}$ is contained in a smallest algebraically closed field $\overline{\mathbb{F}}$, called algebraic closure of $\mathbb{F}$

Ex $\quad \overline{\mathbb{R}} = \mathbb{C}$, $\overline{\mathbb{Q}} = \left\{ \begin{array}{c} \text{algebraic} \\ \text{numbers} \end{array} \right\}$ = sol's to $f(x)=0$, $f \in \mathbb{Q}[x]$

4) $$f_k = \frac{\delta_k}{\delta_{k-1}} \qquad 1 \le k \le n$$

det of submatrix

$$\delta_k = \gcd \left\{ \begin{array}{c} k \times k \text{ minors} \\ \text{of } \; xI_n - A \end{array} \right\}$$

$$A = [T]_{\mathcal{B}}$$

Thus $(f_k)_{k=1}^{n}$ are uniquely determined by T.

<u>Ex</u>. $T = T_A$, $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\delta_1 = 1$

$$\delta_2 = \gcd\left\{ \begin{vmatrix} x-\lambda & 1 \\ 0 & x-\lambda \end{vmatrix}, \cancel{\begin{vmatrix} x-\lambda & 0 \\ 0 & 0 \end{vmatrix}}, \cancel{\begin{vmatrix} x-1 & 0 \\ -\lambda & 0 \end{vmatrix}}, \cancel{\begin{vmatrix} x-\lambda & -1 \\ 0 & 0 \end{vmatrix}}, \begin{vmatrix} x-\lambda & 0 \\ 0 & x-\lambda \end{vmatrix} = (x-\lambda)^2, \underbrace{\begin{vmatrix} -1 & 0 \\ 0 & x-\lambda \end{vmatrix}}_{= -(x-\lambda)} \; \cdots \right\} = x-\lambda$$

$\delta_3 = (x-\lambda)^3$

$f_1 = 1 \quad f_2 = \frac{\delta_2}{\delta_1} = x-\lambda \quad f_3 = \frac{\delta_3}{\delta_1} = (x-\lambda)^2$

<u>Ex</u> $\begin{bmatrix} x-\lambda & -1 & & \\ & x-\lambda & -1 & \\ & & \ddots & -1 \\ & & & x-\lambda \end{bmatrix} 2(x-\lambda)$

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