MATH 510 Practice Problems for Final Exam

1. Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $T : V \to V$ be a linear map of rank $n-1$. Show that there is an ordered basis $\mathcal{B}$ for V such that the matrix $A = [T]_{\mathcal{B}}$ has the form

$$A = \begin{bmatrix} A' & \mathbf{0} \\ u^T & 0 \end{bmatrix}$$

for some invertible $A' \in M_{n-1}$ and some $u \in \mathbb{F}^{n-1}$ (and $\mathbf{0} \in \mathbb{F}^{n-1}$).

2. Let V be vector space. Suppose $P_1, \ldots, P_m : V \to V$ are linear maps satisfying (i) $P_j^2 = P_j$, (ii) $P_j P_k = 0$ for $j \neq k$, (iii) $\sum_j P_j = \mathrm{Id}_V$. (The P_j are said to form a *complete set of mutually orthogonal projections*.)

   (a) Show that $V = \bigoplus_{j=1}^m V_j$, where $V_j = P_j(V)$.

   (b) Conversely, show that if $V = \bigoplus_{j=1}^m V_j$ for some subspaces V_j, then there are linear maps P_j satisfying (i)-(iii) such that $V_j = P(V)$.

   (c) Lastly, if V is an inner product space, show the P_j are Hermitian if and only if $V_j \perp V_k$ for all $j \neq k$. (Here Hermitian means $\langle P_j v, w\rangle = \langle v, P_j w\rangle$ for all $v, w \in V$.)

3. Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $\{v_i\}_{i=1}^n$ and $\{v_i'\}_{i=1}^n$ be two bases for V and let $\{\xi_i\}_{i=1}^n$, $\{\xi_i'\}_{i=1}^n$ be the corresponding dual bases for V^*. Show that in $V^* \otimes V$ we have $\sum_{i=1}^n \xi_i \otimes v_i = \sum_{i=1}^n \xi_i' \otimes v_i'$.

4. Let $\|\cdot\|$ be any vector norm on $M_n(\mathbb{C})$. Define $|||\cdot|||$ by $|||A||| = \max_{\|B\|=1} \|AB\|$. Show that $|||\cdot|||$ is a matrix norm on $M_n(\mathbb{C})$.

5. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $\omega : V \times V \to \mathbb{R}$ be a non-degenerate bilinear form. Show that $\dim V$ is even and there is a basis $\{v_i\}_i$ for V such that

$$\omega(v_i, v_j) = \begin{cases} 1, & i \text{ is odd and } j = i+1 \\ -1 & j \text{ is odd and } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $A, B \in M_n(\mathbb{C})$. If B is nilpotent and commute with A, show that A and $A + B$ have the same characteristic polynomial.

---

5. Pick any basis $\{e_i\}_i$, use real spectral theorem on $A = (\omega(e_i, e_j))_{ij}$.

7. Let $A \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Show the following are equivalent: 1) the minimum polynomial of A is $(x-\lambda)^d$ for some positive integer d. 2) $\lim_{k\to\infty} A^k - \lambda^k I = 0$.

8. If A has invariant factors

$$\{1,\ x+1,\ (x+1)^2(x-2),\ (x+1)^3(x-2)\},$$

find the Jordan normal form of A.

9. If A is a real 8×8 matrix and has minimum polynomial

$$m(x) = (x^2+1)^2(x-3)^2,$$

find the possible real Jordan canonical forms of A.

10. Show that any positive definite matrix P may be factored as $P = LL^*$ where L is a lower-triangular matrix with positive real positive diagonal entries. Show that L is uniquely determined by P.

11. For $A \in M_n(\mathbb{C})$, let $G(A)$ be the union of all (closed) Geršgorin disks of A. Show that $\bigcap_S G(SAS^{-1}) = \sigma(A)$ where the intersection is taken over all nonsingular matrices S. What if we restrict to only unitary S?

12. Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Suppose that all eigenvalues of $A-B$ are non-negative. Show that $\lambda_k(A) \geqslant \lambda_k(B)$ for all $k = 1, 2, \ldots, n$ (where $\lambda_1(X) \leqslant \cdots \leqslant \lambda_n(X)$ denote the eigenvalues of a Hermitian matrix X).

13. Let $A \in M_n$ and $A = V\Sigma W^*$ a singular value decomposition. Let $\operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \Sigma$. Let v_i be the columns of V and w_i the columns of W. Show that $A^*Aw_i = \sigma_i^2 w_i$ and $AA^*v_i = \sigma_i^2 v_i$ for all $i = 1, 2, \ldots, n$. Formulate and prove a converse of this statement.

---

10. Use QR factorization for $P^{1/2}$