MATH 510 Homework 6 **Name:**

- Homework 6 is due March 22 at the beginning of class.

1. **(Corrected March 13)** Let $A \in M_n(\mathbb{C})$. Let $a_1, a_2, \ldots, a_n \in \mathbb{C}^n$ be the columns of A. Prove *Hadamard's inequality*: $|\det A| \leq \prod_{i=1}^{n} \|a_i\|$ with equality if and only if A has orthogonal columns or $a_i = 0$ for some i. *Hint:* Use the QR factorization of A.

2. Let $A \in M_n(\mathbb{R})$ be given. Explain why A is symmetric if and only if A is normal and all its eigenvalues are real.

3. Let $A \in M_n(\mathbb{R})$ be a real normal matrix. If AA^T has n distinct eigenvalues, show that A is symmetric.

4. (a) Show that any real orthogonal matrix has determinant 1 or -1.

   (b) Let $A \in M_n(\mathbb{R})$ be a real orthogonal matrix such that $\det(A) = 1$. Show that there is a continuous function $\gamma : [0, 1] \to O(n)$ such that $\gamma(0) = A$ and $\gamma(1) = I_n$. (That such a function is continuous means that $\gamma(t) = [B_{ij}(t)] \in M_n(\mathbb{R})$ for some continuous functions $B_{ij}(t)$.) (*Hint:* Use the spectral theorem for real normal matrices.)

5. Show that any $B \in M_n(\mathbb{C})$ of the form $B = A^*A$, $A \in M_n(\mathbb{C})$ may be written $B = LL^*$, where $L \in M_n(\mathbb{C})$ is lower triangular and has real non-negative diagonal entries. Explain why this factorization is unique if A is nonsingular.