MATH 403/503 L8

The Correspondence Theorem and the 3rd Isomorphism Theorem.

Q: Given $N \trianglelefteq G$, what are the subgroups of G/N?

Answer is related to inverse images.

First, recall that if $\varphi: G \to H$ is a homomorphism and $K \leq G$, then $\varphi(K) = \operatorname{im} \varphi = \{h \in H : h = \varphi(g), \text{ some } g \in K\}$ is a subgroup of H called the image of K under φ (Proof:

- $e_H = \varphi(e_G) \in \varphi(K)$ since $e_G \in K$, so $\varphi(K) \neq \emptyset$
- if $x, y \in \varphi(K)$ then $x = \varphi(k_1)$, $y = \varphi(k_2)$ for some $k_i \in K$, hence

$$xy^{-1} = \varphi(k_1)\varphi(k_2)^{-1} = \varphi(k_1 k_2^{-1}) \in \varphi(K) \text{ since } K \leq G.$$

So $\varphi(K) \leq H$ by a subgroup criterion. ∎)

Warning: In general, $\varphi(K)$ is not normal in H, even if K is normal in G.

The *inverse image* of a subgroup $K \leq H$ under $\varphi : G \to H$ is

$$\varphi^{-1}(K) = \{ g \in G : \varphi(g) \in K \}.$$

Proposition $\varphi^{-1}(K) \leq G$, and $\ker\varphi \trianglelefteq \varphi^{-1}(K)$

Proof $\varphi(e_G) = e_H \in K$ so $e_G \in \varphi^{-1}(K)$.

If $x, y \in \varphi^{-1}(K)$ then $\varphi(x), \varphi(y) \in K$ so

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in K \text{ since } K \leq H.$$

Thus $xy^{-1} \in \varphi^{-1}(K)$. So $\varphi^{-1}(K) \leq G$.

Since $\ker\varphi \trianglelefteq G$ it suffices to show $\ker\varphi \subseteq \varphi^{-1}(K)$. Let $g \in \ker\varphi$. Then $\varphi(g) = e_H \in K$. So $g \in \varphi^{-1}(K)$. ∎

Remark In general if $K_1 \leq H$, $K_2 \leq H$, $K_1 \neq K_2$ and $\varphi : G \to H$ is a homomorphism, it can happen that $\varphi^{-1}(K_1) = \varphi^{-1}(K_2)$.

For example, if $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_2$, $\varphi(a) = 0 \ \forall a \in \mathbb{Z}_2$ then $\varphi^{-1}(\{0\}) = \varphi^{-1}(\mathbb{Z}_2) = \mathbb{Z}_2$.

However:

<u>Theorem</u> Let $\varphi: G \to H$ be a <u>surjective</u> homomorphism. Then there is a bijective correspondence between subgroups of H and subgroups of G containing $\ker \varphi$:

$$\left\{\begin{array}{c}\text{Subgroups} \\ F \leq G \\ \text{s.t. } \ker\varphi \subseteq F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subgroups} \\ K \leq H\end{array}\right\}$$

$$F \longmapsto \varphi(F)$$

$$\varphi^{-1}(K) \longleftarrow\!\shortmid \; K$$

<u>Proof</u> We show that going over and back is the identity on each side. Start with $K \leq H$. It corresponds to $\varphi^{-1}(K)$ in the LHS. Going back we get $\varphi(\varphi^{-1}(K))$. Since φ is surjective, $K = \varphi(\varphi^{-1}(K))$.

Conversely, let $F \leq G$ with $\ker \varphi \subseteq F$. WTS $\varphi^{-1}(\varphi(F)) = F$. That $F \subseteq \varphi^{-1}(\varphi(F))$ is trivial since $\varphi(f) \in \varphi(F)$ $\forall f \in F$. Let $y \in \varphi^{-1}(\varphi(F))$. Then $\varphi(y) \in \varphi(F)$ so $\varphi(y) = \varphi(f)$, some $f \in F$. $\Rightarrow f^{-1} y \in \ker \varphi \subseteq F \Rightarrow y \in fF = F$. Thus $\varphi^{-1}(\varphi(F)) \subseteq F$.

So $\varphi^{-1}(\varphi(F)) = F$. ∎

<u>Theorem</u> (Correspondence Theorem)

Let G be a group and $N \trianglelefteq G$.

a) There is a bijective correspondence between subgroups F of G containing N, and subgroups of G/N:

$$\left\{ \begin{array}{c} \text{subgroups} \\ F \leq G \\ \text{s.t. } N \subset F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{subgroups} \\ \text{of } G/N \end{array} \right\}$$

$$F \longmapsto F/N =$$

$$\psi^{-1}(K) \longleftarrow\!\!| \; K$$

b) If F_1, F_2 correspond to K_1, K_2 then

$$F_1 \subseteq F_2 \iff K_1 \subseteq K_2$$

$$F_1 \cap F_2 \iff K_1 \cap K_2$$

$$F_1 \trianglelefteq F_2 \iff K_1 \trianglelefteq K_2$$

c) (3^{rd} isomorphism Thm) If $F_1 \trianglelefteq F_2$

$$F_2/F_1 \cong K_2/K_1$$