MATH 403/503 L7

Second Isomorphism Theorem

Notation For subsets X, Y of a group G we put $XY = \{xy : x \in X, y \in Y\}$

Lemma Let G be a group and $H \leq G$, $N \trianglelefteq G$.

a) $HN = NH$

b) $HN \leq G$

Proof a) Let $x \in HN$. Then $x = hn$ for some $h \in H$, $n \in N$. We have

$$x = hn = \underbrace{hnh^{-1}}_{\in N \text{ since } N \trianglelefteq G} h \in NH.$$

Thus $HN \subseteq NH$, conversely, let $y \in NH$. Then $y = mk$ for some $m \in N$, $k \in H$. Now

$$y = mk = k \underbrace{k^{-1}m(k^{-1})^{-1}}_{\in N \text{ since } H \leq G \text{ and } N \trianglelefteq G} \in HN.$$

So $NH \subseteq HN$. Therefore $HN = NH$.

b) $e = e \cdot e \in HN$, so HN is non-empty.

Let $x, y \in HN$, WTS $xy^{-1} \in HN$. We know $x = h_1 n_1$, $y = h_2 n_2$ for some $h_i \in H$, $n_i \in N$.

So $xy^{-1} = h_1 n_1 (h_2 n_2)^{-1} = h_1 n_1 n_2^{-1} h_2^{-1}$.

Let $n = n_1 n_2^{-1}$. Since $N \leq G$, $n \in N$.

Also $h_2^{-1} \in H$ since $H \leq G$. By part a)

$n \cdot h_2^{-1} \in HN$ so $n \cdot h_2^{-1} = \tilde{h} \cdot \tilde{n}$, for some $\tilde{h} \in H$, $\tilde{n} \in N$. Now

$$xy^{-1} = h_1 n h_2^{-1} = \underbrace{h_1 \tilde{h}}_{\in H} \cdot n \in HN.$$

By Subgroup Criterion, $HN \leq G$. ∎

<u>Remark</u> H and N are both subgroups of HN. Since $N \trianglelefteq G$ we also have <u>$N \trianglelefteq HN$</u>. Furthermore, <u>$H \cap N \trianglelefteq H$</u> since

$hnh^{-1} \in H$ when $h \in H$, $n \in H \cap N \subseteq H$, by $H \leq G$

and $hnh^{-1} \in N$ since $n \in N$ and $N \trianglelefteq G$.

<u>Theorem</u> (2^{nd} Isomorphism Theorem)
Let G be a group, $H \leq G$, $N \trianglelefteq G$.
Then

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

<u>Proof</u> Define $\varphi : H \longrightarrow HN/N$ by $\varphi(h) = hN$. <u>φ is a homomorphism</u>:
$\varphi(h_1 h_2) = h_1 h_2 N = (h_1 N)(h_2 N) = \varphi(h_1)\varphi(h_2)$
for all $h_i \in H$.

<u>φ is surjective</u>: Let $x \in HN/N$. Then $x = yN$ for some $y \in HN$. And $y = hn$ for some $h \in H$, $n \in N$. We have

$x = hnN = hN$ since $nN = N$.

So $x = \varphi(h)$.

<u>$\ker \varphi = H \cap N$</u>: Let $h \in \ker \varphi$. Then $\varphi(h) = e$, the identity element in $\frac{HN}{N}$, which is N. $\varphi(h) = N$.
By def. of φ, $\varphi(h) = hN$. Thus $hN = N$ which implies $h \in N$. Since $h \in \ker \varphi \subseteq H$, $h \in N \cap H$.

By the first Isomorphism Theorem, φ induces an isomorphism

$$\bar{\varphi} : \frac{H}{H \cap N} \longrightarrow \frac{HN}{N}$$

$$\bar{\varphi}(h\, H \cap N) = hN.$$

Therefore $\dfrac{H}{H \cap N} \cong \dfrac{HN}{N}$. ∎

<u>Example</u> $G = D_6$, $H = \langle s \rangle = \{e, s\} \leq G$

$N = \langle r^3 \rangle = \{e, r^3\} \leq G$

Since G is generated by $\{s, r\}$ and

$sNs^{-1} = \{ss^{-1}, sr^3s^{-1}\} = \{1, r^3\} = N$

$r^{-3}\cancel{ss^{-1}} = r^3$

$rNr^{-1} = N$ since $rr^3 = r^3 r$

we have $N \trianglelefteq G$.

$HN = \{e, r^3, s, sr^3\}$

$H \cap N = \{e\}$

$$\frac{HN}{N} \cong \frac{H}{H \cap N} \cong H \, (\cong \mathbb{Z}_2)$$

Example. $G = Q_8$ $\quad H = \{1, i, -1, -i\} \leq Q_8$

$N = \{1, -1\} \trianglelefteq Q_8$

(in fact, since $|Q_8/H| = 2$, here $H \trianglelefteq Q_8$ too)

$HN = H$ since $N \subseteq H$

$H \cap N = N.$

$$\frac{H}{N} = \frac{HN}{N} \cong \frac{H}{H \cap N} = \frac{H}{N}$$ trivial in this case!

More generally, when $N \subseteq H$, the conclusion is trivial.

Example. $G = \mathbb{Z}_{24}$ $\quad H = \langle 4 \rangle = \{0, 4, 8, 12, 16, 20\}$

$N = \langle 6 \rangle = \{0, 6, 12, 18\}$

$H + N = \langle \gcd(4,6) \rangle = \langle 2 \rangle = \{0, 2, 4, \ldots, 22\}$

$H \cap N = \langle 12 \rangle = \{0, 12\}$

By 2nd Isom. Thm:

$$\frac{\langle 2 \rangle}{\langle 6 \rangle} \cong \frac{\langle 4 \rangle}{\langle 12 \rangle}$$

(The order of each side is $\frac{12}{4} = \frac{6}{2}$ OK.)