MATH 403/503 L5

The *symmetric group* S_n is the set of all bijections $\sigma : \{1, 2, .., n\} \to \{1, 2, .., n\}$. Elements of S_n are called *permutations*. Any $\sigma \in S_n$ can be written explicitly using *two-line notation*:

If $\sigma(1) = i_1$, $\sigma(2) = i_2$, ..., $\sigma(n) = i_n$ we write

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & & i_n \end{pmatrix}$$

Ex In S_4 we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

(Recall $(f \circ g)(x) = f(g(x))$ so for ex $(\sigma \circ \tau)(4) = \sigma(\tau(4))$.)

and $\begin{pmatrix} 1\uparrow & 2\uparrow & 3\uparrow & 4\uparrow \\ 2 & 3 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

(Read bottom to top)

A *cycle* $\sigma \in S_n$ is a permutation of the form $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, ..., $\sigma(a_k) = a_1$ and $\sigma(b) = b$ if $b \notin \{a_1, a_2, .., a_k\}$.

We write $\sigma = (a_1 \; a_2 \; \cdots a_k)$.

- Two cycles $(a_1 a_2 \cdots a_k)$, $(b_1 b_2 \cdots b_\ell)$ are <u>disjoint</u> if $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_\ell\} = \emptyset$.
- k is the <u>length</u> of $(a_1 a_2 \cdots a_k)$
- Length 2 cycles $(i\ j)$ are <u>transpositions</u>

<u>Ex</u> $(1\,3\,5)$, $(2\,4)$ are disjoint.

<u>Note</u>: $(a_1\, a_2 \cdots a_k) = (a_1\, a_2)(a_2\, a_3) \cdots (a_{k-1}\, a_k)$

<u>Any length k cycle is a product of $k-1$ transpositions</u>

<u>Ex</u> $(1\,2)$ length 2, 1 transp.

$(1\,2\,3) = (1\,2)(2\,3)$ length 3, 2 transp.

$(7\,3\,2\,8\,1) = (7\,3)(3\,2)(2\,8)(8\,1)$

<u>Thm</u> 1 a) Every permutation is a product of pairwise disjoint cycles.

b) Every permutation is a product of transpositions.

c) For given $\sigma \in S_n$, the number of transpositions in any factorization of σ into a product of transpositions is either always even or always odd.

Ex $(1\,3) = (1\,2)(2\,3)(1\,2) = (2\,3)(1\,2)(2\,3)$
but according to Thm 1(c) we can never write $(1\,3)$ as a product of an even number of transpositions.

Def $\sigma \in S_n$ is even if it can be written as a product of an even number of transpositions. Otherwise σ is called odd.

~~Def~~ The sign homomorphism is defined by

$$\operatorname{sgn} : S_n \to \{\pm 1\}$$

$$\operatorname{sgn} \sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Thm sgn is a homomorphism

Pf $\rho = \tau_1 \cdots \tau_k$, $\pi = \sigma_1 \cdots \sigma_\ell$
chosen factorizations of $\rho, \pi \in S_n$ into transpositions τ_i , σ_j . Note
$\operatorname{sgn} \rho = (-1)^k$ $\quad \operatorname{sgn} \pi = (-1)^\ell$ and

$$\operatorname{sgn}(\rho \circ \pi) = \operatorname{sgn}(\tau_1 \cdots \tau_k \sigma_1 \cdots \sigma_\ell) = (-1)^{k+\ell}$$

Thus $\operatorname{sgn}(\rho \circ \pi) = (\operatorname{sgn} \rho)(\operatorname{sgn} \pi)$ ∎

<u>Cor</u> $\ker(\mathrm{sgn}) \trianglelefteq S_n$

<u>Notation</u>

$A_n := \ker(\mathrm{sgn}) = \{\sigma \in S_n \mid \sigma \text{ is even}\}$

This is the <u>alternating group</u>.

<u>Note</u> For $n > 1$, sgn is surjective since $\mathrm{sgn}(12) = -1$.

By the First Isomorphism Theorem,

$\overline{\mathrm{sgn}} : \dfrac{S_n}{A_n} \longrightarrow \{\pm 1\}$

is an isomorphism. Indeed:

$S_n/A_n = \{ A_n , \underset{\{\text{odd permutations}\}}{\underset{\parallel}{(12)A_n}} \}$

S_n/A_n	A_n	$(12)A_n$
A_n	A_n	$(12)A_n$
$(12)A_n$	$(12)A_n$	A_n

Show that

$$\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$$

① Define $\varphi : \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_5$

$\varphi(a) = ([a]_3, [a]_5)$.

- φ hom.
- ~~φ surj~~
- $\operatorname{Ker} \varphi = 15\mathbb{Z}$.
- Iso Thm $\rightsquigarrow$ inj. $\bar{\varphi} : \mathbb{Z}_{15} \to \mathbb{Z}_3 \times \mathbb{Z}_5$

Let $\varphi : G \to H$

Then $\bar{\varphi} : G/\operatorname{ker}\varphi \to H$

is injective.