MATH 403/503 L4

<u>Def</u> A map $\varphi : G \to H$ is a

1) <u>homomorphism</u> if $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$
2) <u>isomorphism</u> if $\exists$ homom. $\psi : H \to G$ st $\psi \circ \varphi = Id_G$, $\varphi \circ \psi = Id_H$
3) <u>automorphism</u> if φ is an isomorphism and $H = G$.

<u>Lem</u> φ is an isomorphism iff it is a bijective homomorphism.

<u>Pf</u> Exercise.

<u>Def</u> The <u>kernel</u> of a homomorphism $\varphi : G \to H$ is

$$\operatorname{Ker} \varphi = \{ g \in G ; \varphi(g) = e_H \}$$

<u>Ex.</u> Recall $\varphi : \mathbb{R} \to GL_2(\mathbb{R})$

$$\varphi(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\operatorname{Ker} \alpha = 2\pi\mathbb{Z}$.

<u>Theorem</u> Let G, H be groups and let $\varphi: G \to H$ be a homomorphism.

Then

1) $\ker\varphi \trianglelefteq G$

2) φ is injective $\iff \ker\varphi = \{e_G\}$

<u>Proof</u> 1) Let $x, y \in \ker\varphi$. Then

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = e_G\, e_G^{-1} = e_G$$

$\Rightarrow xy^{-1} \in \ker\varphi$. Also, $\varphi(e_G) = e_H$. So $\ker\varphi \neq \emptyset$. By subgrp criterion, $\ker\varphi \leq G$.

Now let $g \in G$, $x \in \ker\varphi$. WTS $gxg^{-1} \in \ker\varphi$. We have

$$\varphi(gxg^{-1}) = \varphi(g)\underbrace{\varphi(x)}_{e_H}\varphi(g)^{-1} = \cancel{\varphi(g)\varphi(g)^{-1}} = e_H.$$

Thus $\ker\varphi \trianglelefteq G$.

2) $(\Rightarrow)$: Suppose $x \in \ker\varphi$. Then $\varphi(x) = e_H$. But also $\varphi(e_G) = e_H$. Since φ is injective, $x = e_G$. Therefore $\ker\varphi = \{e_G\}$.

$(\Leftarrow)$: Suppose $\varphi(x) = \varphi(y)$. Then $\varphi(x)\varphi(y)^{-1} = e_H$ so $e_H = \varphi(x)\varphi(y^{-1}) = \varphi(xy^{-1})$. So $xy^{-1} \in \ker\varphi$. By assumption $\ker\varphi = \{e_G\}$. So $xy^{-1} = e_G$. Therefore $x = y$. So φ is injective. $\blacksquare$

A rule such as

For $a \in \mathbb{R}_{>0}$ let $f(a) = \begin{pmatrix} \text{a solution} \\ \text{to } x^2 = a \end{pmatrix}$

does not give a function since
$f(2) = \sqrt{2}$ but also $f(2) = -\sqrt{2}$

We say f is not <u>well-defined</u>.

For a rule f to give a function we must have that if $a = b$ then $f(a) = f(b)$.

<u>Theorem</u> 2 Let $\varphi: G \to H$ be a homomorphism of groups G, H. Let $N \trianglelefteq G$. Consider the rule $\bar{\varphi}: G/N \to H$

$$\bar{\varphi}(gN) = \varphi(g).$$

1) The rule gives a well-defined function $\bar{\varphi}$ iff $N \subseteq \ker \varphi$. Then $\bar{\varphi}$ is a homomorphism

2) If $N \leq \ker \varphi$ then $\bar{\varphi}$ is injective iff $N = \ker \varphi$.

3) If $N = \ker \varphi$ then $\bar{\varphi}: G/N \to \varphi(G)$ is an isomorphism, where

$$\varphi(G) = \operatorname{im} \varphi = \{h \in H : h = \varphi(g) \text{ for some } g \in G\}$$

Remark: Part 3) is known as the First Isomorphism Theorem.

Proof 1) $\bar{\varphi}$ is well-defined iff

$$gN = g'N \implies \varphi(g) = \varphi(g')$$

This is equivalent to

$$(g')^{-1}gN = N \implies \varphi((g')^{-1}g) = e_H$$

or

$$(g')^{-1}g \in N \implies (g')^{-1}g \in \ker\varphi$$

In other words, $N \subseteq \ker \varphi$.

Suppose $N \subseteq \ker\varphi$. Then

$$\bar{\varphi}(gN\, g'N) = \bar{\varphi}(gg'N) = \varphi(gg') =$$

$$= \varphi(g)\varphi(g') = \bar{\varphi}(gN)\,\bar{\varphi}(g'N)$$

for all $gN, g'N \in G/N$. So $\bar{\varphi}$ is a homomorphism.

2) $\ker\bar{\varphi} = \{gN : \bar{\varphi}(gN) = e_H\} =$

$$= \{gN : \varphi(g) = e_H\} = \{gN : g \in \ker\varphi\}$$

$$\overset{N=\ker\varphi}{=} \{gN : g \in N\} = \{N\} = \{e_{G/N}\}$$

By Theorem 1, $\bar{\varphi}$ is injective, if $N = \ker\varphi$

Conversely, if $\bar{\varphi}$ is injective, then $\ker \bar{\varphi} = \{N\}$ by Theorem 1, so every $g \in \ker \varphi$ must give $gN = N$. That is $\ker \varphi \subseteq N$. We already assume $\ker \varphi \supseteq N$. So $\ker \varphi = N$.

3) If $N = \ker \varphi$ then by 1), 2) $\bar{\varphi}$ is well-defined & injective. The image of $\bar{\varphi}$ is $\bar{\varphi}(G/N) = \{\bar{\varphi}(gN) : g \in G\} = \{\varphi(g) : g \in G\} = \varphi(G)$. Therefore $\bar{\varphi}$ defines a bijective homomorphism

$$\bar{\varphi} : G/_{\ker \varphi} \to \varphi(G)$$

By Lemma, $\bar{\varphi}$ is an isomorphism.

Example $\varphi : \mathbb{R} \to GL_2(\mathbb{R})$

$$\varphi(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\ker \varphi = 2\pi\mathbb{Z}$. One can show:

$$\operatorname{im} \varphi = \left\{ A \in GL_2(\mathbb{R}) : \begin{array}{l} A \cdot A^T = I \text{ and} \\ \det(A) = 1 \end{array} \right\}$$

$= SO_2(\mathbb{R})$ special orthogonal group.

By the First Isomorphism Theorem,

$$\mathbb{R}/_{2\pi\mathbb{Z}} \cong SO_2(\mathbb{R}).$$

Example

$$\varphi : S_n \longrightarrow \{\pm 1\}$$

$$\varphi(\sigma) = \operatorname{sgn} \sigma$$

$\ker S_n = A_n$ alternating group.

So by 1st Iso Th.

$$S_n/_{A_n} \cong \{\pm 1\}, \quad n > 1$$

(For $n=1$, φ is not surj.)