403/503 L38

<u>Thm</u> $x \in \mathbb{R}_c$ ~~then~~ iff

$\exists\ x_1, \dots, x_n \in \mathbb{R}$ s.t.

$x_1^2 \in \mathbb{Q}$, $x_j^2 \in \mathbb{Q}(x_1, \dots, x_{j-1})$ $2 \le j \le n$

and $x \in \mathbb{Q}(x_1, \dots, x_n)$.

<u>proof</u>. $(\Leftarrow)$: It suffices to show

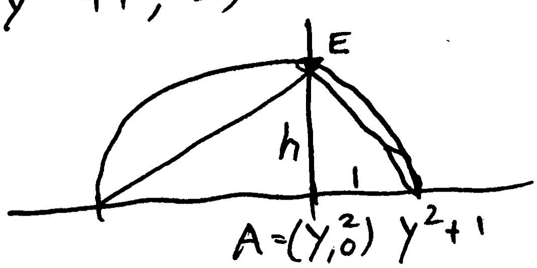
$$y^2 \in \mathbb{R}_c \Rightarrow y \in \mathbb{R}_c \qquad (*)$$

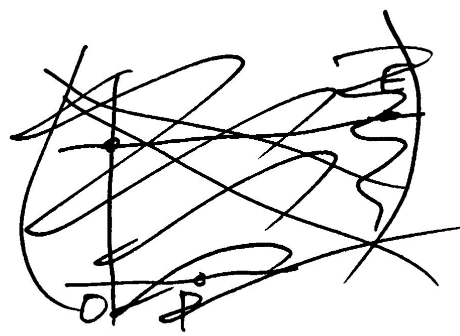
Indeed, then $x_1 \in \mathbb{R}_c$ ~~and~~, hence $\mathbb{Q}(x_1) \subseteq \mathbb{R}_c$ Thus $x_2^2 \in \mathbb{Q}(x_1) \subseteq \mathbb{R}_c \Rightarrow x_2 \in \mathbb{R}_c$

etc.

To prove $(*)$, note $y^2 \in \mathbb{R}_c \Rightarrow y^2+1 \in \mathbb{R}_c$

$\Rightarrow (y^2+1, 0) \in Con$





Consider $C\left(\left(\frac{y^2+1}{2}, 0\right), 0\right)$, L line through y^2, $\perp$ to $\overline{OP}$. Let E be pt of $\cap$.

Then $\frac{h}{1} = \frac{y^2}{h}$ so $|AE| = y$

~~Draw~~ $\Rightarrow E = (y^2, y) \in Con \Rightarrow y \in \mathbb{R}_c$.

$(\Rightarrow)$: Let $\tilde{\mathbb{R}}_c$ be the set

$$\left\{ x \in \mathbb{R} \;\middle|\; \exists\, x_1, \dots, x_n \in \mathbb{R} : \begin{array}{l} x_1^2 \in \mathbb{Q} \\ x_j^2 \in \mathbb{Q}(x_1, \dots, x_{j-1}) \\ x \in \mathbb{Q}(x_1, \dots, x_n) \end{array} \right\}$$

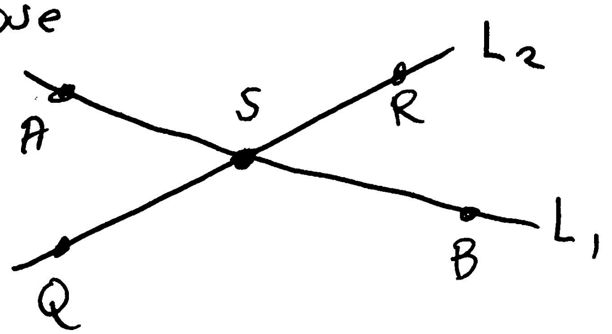
WTS $\mathbb{R}_c \subseteq \tilde{\mathbb{R}}_c$. Note: $\tilde{\mathbb{R}}_c$ subfield of $\mathbb{R}$.

A seq $(x_1, \dots, x_n) \in \mathbb{R}$ is a <u>square-root sequence</u> if $x_1^2 \in \mathbb{Q}$, $x_j^2 \in \mathbb{Q}(x_1, \dots, x_{j-1})$ $2 \leq j \leq n$.

<u>Note</u>: (x_i), (y_j) SRS $\Rightarrow$ $(x_1, \dots, x_n, y_1, \dots, y_m)$ SRS

Clearly $0, 1 \in \tilde{\mathbb{R}}_c$. (Trivial)

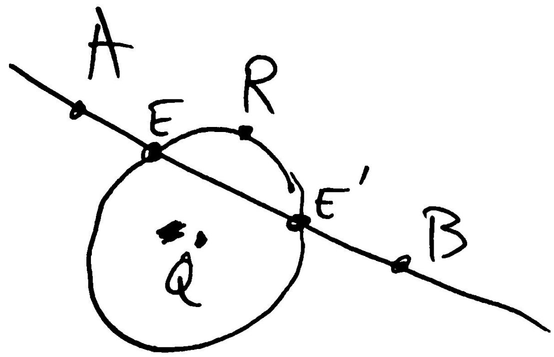
Suppose



where $A = (a_1, a_2)$ etc and all $a_i\, b_i\, r_i\, q_i \in \mathbb{Q}(x_1, \dots, x_n)$

By two-point formula

$$\begin{array}{l} L_1: \\ L_2: \end{array} \begin{cases} ax + by + c = 0 \\ dx + ey + f = 0 \end{cases} \Rightarrow \text{coords } (x, y) \text{ of } S \text{ also } \in \mathbb{Q}(x_1, \dots, x_n).$$



Suppose A, B, Q, R have coords in $\tilde{\mathbb{R}}_c$. Then

$$\begin{cases} (x-q_1)^2 + (y-q_2)^2 = (r_1-q_1)^2 + (r_2-q_2)^2 \\ \left(\frac{y-b_2}{x-b_1} = \frac{a_2-b_2}{a_1-b_1} \right) \\ ax + by + c = 0 \end{cases}$$

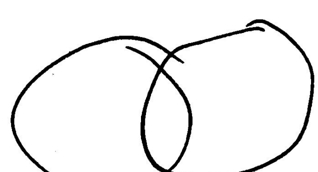
has coeffs in $\tilde{\mathbb{R}}_c$, say $\in \mathbb{Q}(x_1, \dots, x_n)$.

Solving for y (if $b \neq 0$) & subst.

we get $(x-q_1)^2 \in \mathbb{Q}(x_1, \dots, x_n)$

Hence $x \in \mathbb{Q}(x_1, \dots, x_n, x_{n+1})$, $x_{n+1} := (x-q_1)^2$

So E, E' have coords in $\tilde{\mathbb{R}}_c$.

Similarly 

$$\begin{cases} x^2 + y^2 + ax + by + c = 0 \\ x^2 + y^2 + a'x + b'y + c' = 0 \end{cases}$$

all coeff $\in \tilde{\mathbb{R}}_c$
$\Rightarrow x, y \in \tilde{\mathbb{R}}_c$.

This proves $\mathbb{R}_c \subseteq \tilde{\mathbb{R}}_c$.

$\blacksquare$

Cor If $x \in \mathbb{R}_c$ then

$$[\mathbb{Q}(x) : \mathbb{Q}] = 2^r$$

Pf ~~$\mathbb{Q}(x) \subset \mathbb{Q}(x_1, \dots, x_n$~~

$$\mathbb{Q} \underset{1 \text{ or } 2}{\subset} \mathbb{Q}(x_1) \underset{\le 2}{\subseteq} \dots \underset{\le 2}{\subseteq} \mathbb{Q}(x_1, \dots, x_n) \ni x$$

$$\Rightarrow [\mathbb{Q}(x_1, \dots, x_n) : \mathbb{Q}] = 2^r$$

$$\Rightarrow [\mathbb{Q}(x) : \mathbb{Q}] \mid [\mathbb{Q}(x_1, \dots, x_n) : \mathbb{Q}]$$

$$\Rightarrow \quad [\mathbb{Q}(x) : \mathbb{Q}] = 2^s$$

Trisecting angle.

$$\theta = \frac{\pi}{3}$$

$$\left[\mathbb{Q}\left(\cos\left(\frac{\pi}{9}\right)\right) : \mathbb{Q}\right] = 3$$

(Exercise)

Regular heptagon

p-gon p prime

$\cos \frac{2\pi}{p}$

$$m_{\zeta_p, \mathbb{Q}}(x) = x^{p-1} + \dots + 1$$

$p - 1 = 2^r$ ~~except p a Fermat prime~~

(In fact if $2^r + 1$ prime
then $r = 2^k$)

$\Leftrightarrow$ p Fermat prime