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We let Con(S) denote all points in the plane that can be obtained from S using ruler & compass.

We let $\mathbb{R}_c$ be the set of real numbers $a \in \mathbb{R}$ such that

$\exists b \in \mathbb{R}: \ (a,b) \in Con(\{O=(0,0), \ P=(1,0)\})$

or $\exists b \in \mathbb{R}: \ (b,a) \in$ —— " ——

That is, with $S = \{(0,0), (1,0)\}$

$$\mathbb{R}_c = \pi_1(S) \cup \pi_2(S)$$

$\pi_1: \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto x$

$\pi_2: \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto y.$

Lem: If $a, b \in \mathbb{R}_c$ then $a-b \in \mathbb{R}_c$.

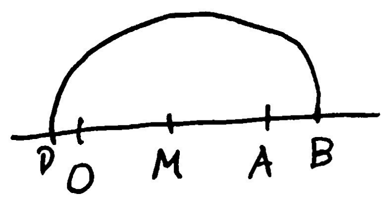
Proof Let M be the midpoint on OA.

Thus $M = (\frac{1}{2}c, 0)$, $A = (a, 0)$, $B = (b, 0)$.

If $B \neq M$: $C(M, B)$ intersects $\overline{OA}$ in B and $D = (\frac{1}{2}a \pm |MB|, 0)$.

We have $|MB| = |\frac{1}{2}a - b|$.

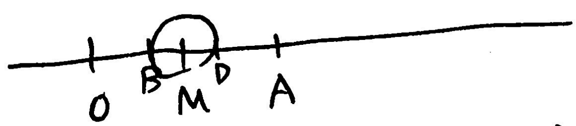
If $\frac{1}{2}a < b$, then



M is to the left of B

and $D = (\frac{a}{2} - (b - \frac{a}{2}), 0) = (a-b, 0)$

if $\frac{1}{2}a > b$ then M is to the right of B



and $D = (\frac{a}{2} + (\frac{a}{2} - b), 0) = (a-b, 0)$.

If $B = M$ then $b = \frac{1}{2}a$ so $a - b = \frac{1}{2}a$

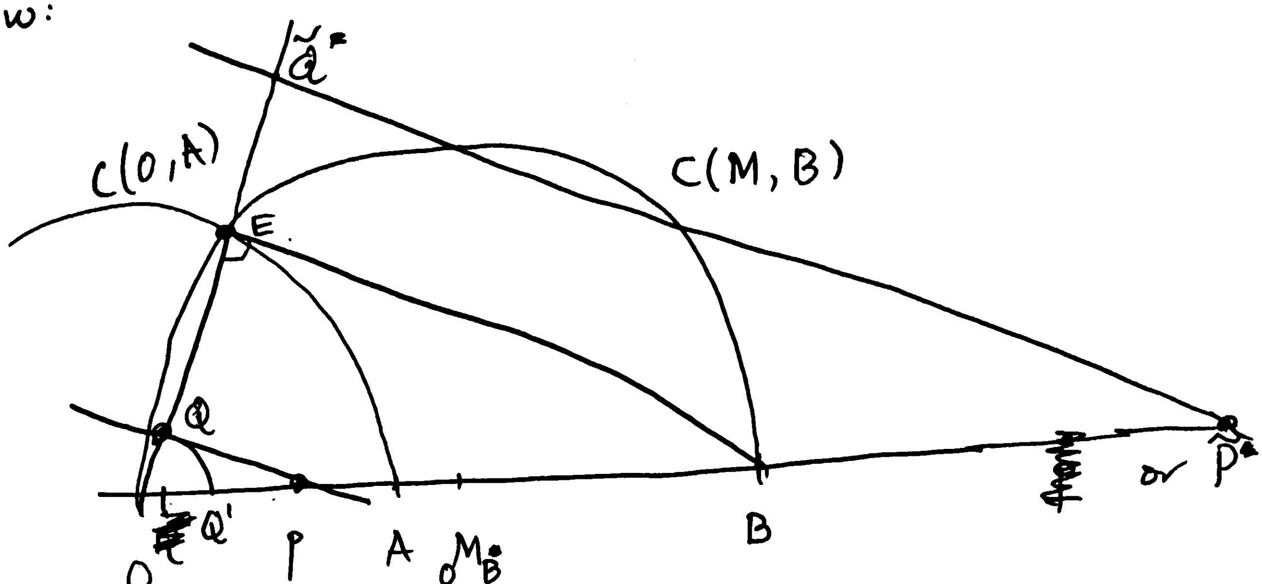
i.e. $M = (\frac{1}{2}a, 0) = (a-b, 0)$

Thus $a - b \in \mathbb{R}_c$.

Lem If $a, b \in \mathbb{R}_c$, $b \neq 0$, then $a/b \in \mathbb{R}_c$.

Proof As before WLOG $A = (a,0)$, $B = (b,0)$ belong to $Con(S)$, $S = \{(0,0), (1,0)\}$.

Now:



① $C(O,A)$, $C(M_B, B)$, E = pt of $\cap$

② Draw line L through P, parallel to $\overline{EB}$

Let Q be pt of $\cap$ between L & $\overline{OE}$.

③ $Q' = \overline{OP} \cap C(O,Q)$

Then:

$$|OQ'| = \frac{|OQ|}{|OP|} = \frac{|OE|}{|OB|} = \frac{a}{b}$$

So $Q' = (\frac{a}{b}, 0)$ but $Q' \in Con(S)$

So $\frac{a}{b} \in \mathbb{R}_c$

∎

Th. $\mathbb{R}_c$ is a subfield of $\mathbb{R}$ (containing $\mathbb{Q}$)

<u>Proof</u> $(0,0), (1,0) \in S \subseteq Con(S)$
so $0, 1 \in \mathbb{R}_c$ so $\mathbb{R}_c \neq \emptyset, \{0\}$
By lemmas, $\mathbb{R}_c$ is closed under subtraction and division.
Thus $\mathbb{R}_c$ is a subfield of $\mathbb{R}$.
(Every subfield of $\mathbb{R}$ contains $\mathbb{Q}$.)

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