403/503 L36 ~~505 ?2?~~

Constructions with Straightedge & Compass

Geometers of ancient Greece over 2000 years ago considered geometric constructions using only a straightedge (unmarked ruler) and compass.

With a straightedge we can draw the line through two previously constructed points.

With a compass we can draw a circle whose center is a constructed point and whose circumference passes through a constructed point.

x The Greeks succeeded in constructing

- regular n-gons, for n=3,4,5,6
- a square twice the area of a given square (doubling the square)
- an angle half that of a given one (bisecting an angle)

ut, they were unable to construct

- regular heptagon (n=7)
- the side of a cube twice the volume of a ~~given~~ cube with given side (doubling the cube)
- an angle $\frac{1}{3}$ that of a given one (trisecting an angle)

Only in the 19^{th} century it was proved these cannot be constructed (with straightedge & compass).

Anectode Doubling the cube is called the Delian problem: The legend says there was a terrible plague in Athens 2500 years ago. The Athens consulted the oracle at Delos about what to do. The oracle told them to double the cubical altar to Apollo. They doubled its side (giving ×8 volume). But the plague continued. It was surmised that perhaps the oracle had meant they should double the volume. At this point it became a pressing matter to construct $\sqrt[3]{2}$ (equivalent to doubling the cube).

~~3.3~~ Straightedge and Compass Constructions

<u>Def</u> Let $S \subseteq \mathbb{R}^2$. The set $Con(S)$ of points <u>obtained from S by straightedge and compass constructions</u> is defined recursively as follows:

(1) $S \subseteq Con(S)$

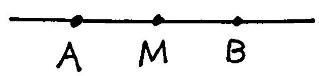
(2) If $P, Q, A, B \in Con(S)$ where $P \neq Q$ and $A \neq B$ then $X \in Con(S)$ where X is any point of intersection of
  (i) the lines $\overline{PQ}$ & $\overline{AB}$, provided $\overline{PQ} \nparallel \overline{AB}$
  (ii) the line $\overline{PQ}$ and the circle centered at A through B
  (iii) the circle through B centered at A and the circle through Q centered at P.

WLOG we will assume $O=(0,0)$ and $P=(1,0)$ always belong to S.

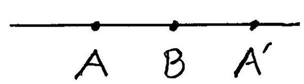
Auxilliary Constructions

(1) Midpoint constructions

(a) If $A, B \in Con(S)$, $A \neq B$
then the midpoint $M \in Con(S)$



(b) If $A, B \in Con(S)$, $A \neq B$
then the point A' on $\overline{AB}$
such that $|AB| = |BA'|$ belongs
to $Con(S)$

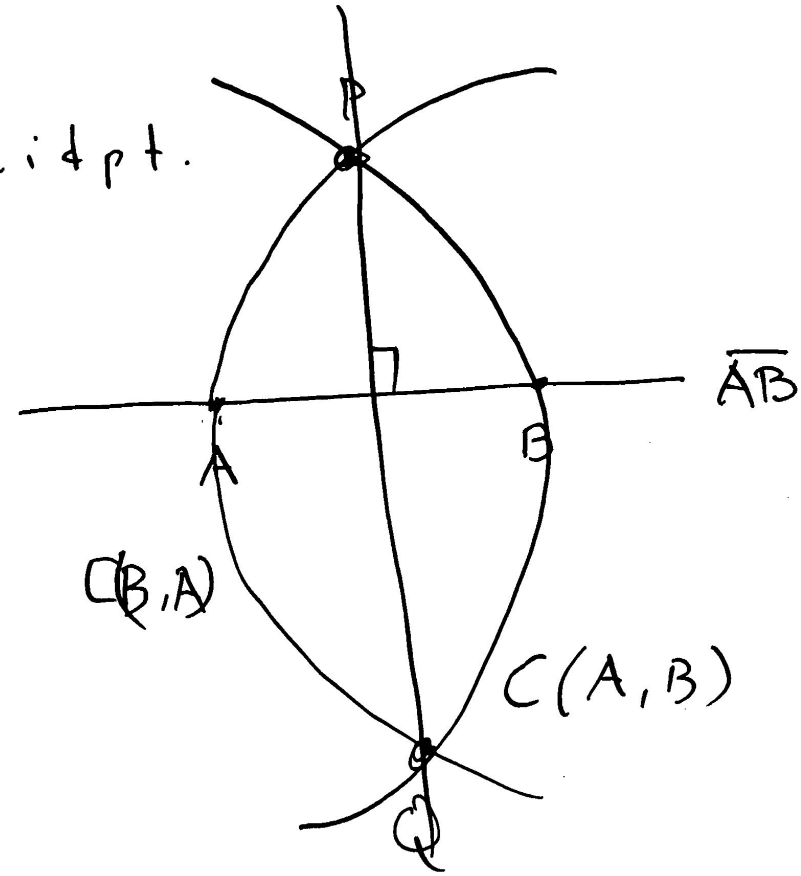


Proof: Exercise to the reader.

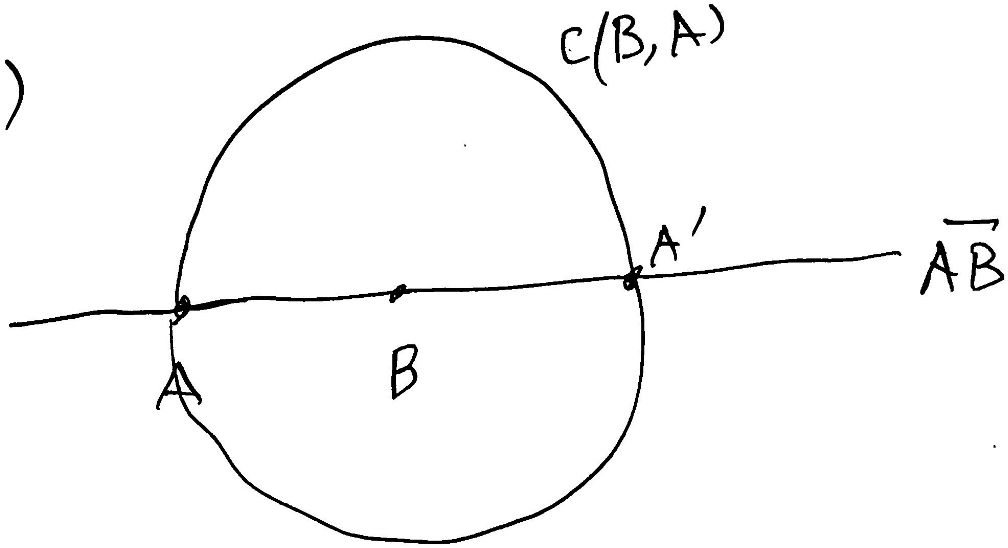
Notation: $M = \frac{1}{2} AB$

Midpt.

(a)


P
$\overleftrightarrow{AB}$
A
B
C(B,A)
C(A,B)
Q

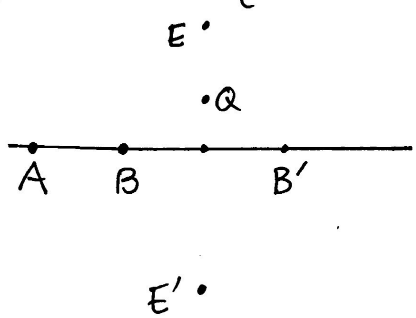
(b)


C(B,A)
A′
$\overleftrightarrow{AB}$
A
B

(2) Orthogonal line:

If $A, B, Q \in Con(S)$, $A \neq B$, then $\exists D \in Con(S)$ such that $\overline{DQ} \perp \overline{AB}$

Proof Replacing B by $\frac{1}{2}AB$ if necessary,
WLOG $C(Q,B) \cap \overline{AB} = \{B, B'\}$, $B' \neq B$.
Let $\{E, E'\} = C(B,B') \cap C(B',B)$. Then
any $D \in \{E, E', \frac{1}{2}BB'\} \setminus \{Q\}$ works.
QED



(3) Parallel line:

If $A, B, Q \in Con(S)$, $A \neq B$, then $\exists D \in Con(S)$ such that $\overline{DQ} \parallel \overline{AB}$.

Proof Let $E \in Con(S)$ be such that $\overline{QE} \perp \overline{AB}$
WLOG $E \in \overline{AB}$, $Q \notin \overline{AB}$. Let $D \in Con(S)$ be
such that $\overline{DQ} \perp \overline{QE}$. Then $\overline{DQ} \parallel \overline{AB}$. QED.