403/503 L19

Basic Properties of Rings.

Proposition Let R be a ring and $a, b \in R$. Then

1) $a \cdot 0 = 0 = 0 \cdot a$
2) $a(-b) = -(ab) = (-a)b$
3) $(-a)(-b) = ab.$

Proof 1) $a0 + a0 = a(0+0) = a0$

By cancellation law in a group we get $a0 = 0$. Similarly $0a = 0$.

2) $a(-b) + ab = a((-b)+b) = a0 \overset{1)}{=} 0$

Since $(R,+)$ is a group we get

$a(-b) = -(ab)$.

Similarly $(-a)b = -(ab)$.

3) $(-a)(-b) \overset{2)}{=} -(a(-b)) \overset{2)}{=} -(-(ab)) = ab.$

$\blacksquare$

Subrings

Def A ring S is a <u>subring</u> of a ring R if

1) S is a subset of R
2) the inclusion map $i : S \to R$, $i(x) = x$, is a ring homomorphism.

Note 2) may also be stated: S is a ring using the same operations as in R.

Ex $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ is a chain of subrings.

Note We will always assume rings have identity 1_R. Also, by definition of a ring homomorphism, if S is a subring of R then $1_S \underset{\text{by def. of } i}{=} i(1_S) \underset{\text{by def of ring homom.}}{=} 1_R$.

Ex For $n > 1$, $n\mathbb{Z}$ is NOT considered a subring of $\mathbb{Z}$.

Proposition (Subring criterion)

A subset S of a ring R becomes a subring of R if and only if

i) $1_R \in S$

ii) if $x, y \in S$ then $x \cdot y \in S$
product in R

iii) if $x, y \in S$ then $\underbrace{x - y}_{x+(-y)} \in S$
negation in R
addition in R

Proof By i), $S \neq \emptyset$ and by iii) S is closed under subtraction. Thus $(S,+)$ is a subgroup of $(R,+)$.
By i) and ii), S is a submonoid of $(R,\cdot)$. Distributive laws hold in S, since they hold in R.
This ~~means~~ means that all axioms of a ring hold for S when we use the operations $+$, $\cdot$ from R.

Example

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Show T is a subring of $M_2(\mathbb{R})$.

Sol. $1_{M_2(\mathbb{R})} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in T$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ 0 & c-z \end{pmatrix} \in T$$

so T closed under subtraction.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay+bz \\ 0 & cz \end{pmatrix} \in T.$$

so T closed under mult.

Zero-Divisors and Integral Domains.

Def Let R be commutative ring. $r \in R$ is a zero-divisor if

i) $r \neq 0$

ii) $\exists s \neq 0 \ : rs = 0$

Example $[2]$ is a zero-divisor in the ring $\mathbb{Z}_6$ because $[2] \neq [0]$ and $[2] \cdot [3] = [0]$ and $[3] \neq 0$.

Note that this also shows that $[3]$ is a zero-divisor in $\mathbb{Z}_6$.

Def A commutative ring without zero-divisors is an integral domain.

Proposition Any subring of a field is an integral domain.

Proof Let $r \in R$, $r \neq 0$, where R is a subring of a field $\mathbb{F}$. Since $r \neq 0$, r is a unit in $\mathbb{F}$. Suppose $rs = 0$, for some $s \in R$. Then $0 = r^{-1}(rs) = (r^{-1}r)s = 1s = s$, hence $s = 0$.

This shows that r is not a zero-divisor. Thus R is an integral domain. ∎