Thm (Sylow's Thm)

Let G grp, $|G| = p^\alpha m$, p prime, $p \nmid m$

Then

1) $Syl_p(G) \neq \emptyset$. ~~That~~ is,

There exists a subgroup of G of order p^α.

2) If P is a Sylow p-subgroup of G and Q is any p-subgroup of G then $\exists g \in G$: $Q \leq gPg^{-1}$.

In particular any two Sylow p-subgrps of G are conjugate.

3) The number of Sylow p-subgroups is of the form $1+kp$, $k \in \mathbb{Z}, k \geq 0$

That is $n_p \equiv 1 \mod p$

Furthermore, n_p is the index in G of the normalizer $N_G(P)$ for any Sylow p-subgroup P, hence $n_p \mid m$.

Pf of Sylow Th (1)

Induction on $|G|$

$|G| = 1$ trivial

Assume $Syl_p(H) \neq \emptyset \; \forall$ groups H, $|H| < |G|$.

If $p \mid |Z(G)|$ then by Cauchy's Thm for abelian groups, $Z(G)$ has a subgrp N of order p. Then $|G/N| < |G|$ so G/N has a Sylow p-subgrp $\overline{P}$ of order $p^{\alpha-1}$.

Let P be the subgroup of G containing N such that $P/N = \overline{P}$. (corresp. thm)

Then $|P| = |P/N| \cdot |N| = p^{\alpha}$ so P is a Sylow p-subgrp of G.

If $p \nmid |Z(G)|$ consider class eq

$$|G| = |Z(G)| + \sum_{i=1}^{n} |G : C_G(g_i)|$$

if $p \mid |G : C_G(g_i)| \; \forall i \Rightarrow p \mid |Z(G)|$ contrad.

So $\exists i$: $p \nmid |G : C_G(g_i)|$

Let $H = C_G(g_i)$. Then

$|H| = p^\alpha k$, $p \nmid k$ (same α as for G)

Since $g_i \notin Z(G)$, $|H| < |G|$

Ind $\Rightarrow$ H has Sylow p-subgrp P

$P \leq H \leq G \Rightarrow P$ Sylow p-subgrp of G

(2) Let $S = \{P_1, \ldots, P_r\}$ be the set of conjugates of P

Let Q any p-subgrp of G

$S = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_s$ orbits w.r.t. Q

~~where~~ $\Rightarrow r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$

By numbering WLOG $P_1 \in \mathcal{O}_1$, $P_2 \in \mathcal{O}_2$ etc.

$|\mathcal{O}_i| = |Q : N_Q(P_i)|$, $N_Q(P_i) = N_G(P_i) \cap Q$

But by Lemma 19, this is $P_i \cap Q$

$|\mathcal{O}_i| = |Q : P_i \cap Q|$

In particular for $Q = P_1$

$|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$

$\forall i > 1 \quad P_1 \neq P_i$ so $P_1 \cap P_i < P_i$

so $|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1, \forall i > 1$

P_1 p-group $\Rightarrow p \,\big|\, |\mathcal{O}_i| \quad \forall i > 1$

$\Rightarrow r = |\mathcal{O}_1| + \left(|\mathcal{O}_2| + \dots + |\mathcal{O}_s|\right) \equiv 1 \pmod{p}$

<u>pf of (2)</u> Let Q any p-subgrp of G

if $\forall g: Q \not\leq gPg^{-1}$

then $Q \cap P_i < Q \quad \forall i$

So $|\mathcal{O}_i| > 1 \quad \forall i \Rightarrow p \mid r$

contrad. $r \equiv 1 \pmod{p}$

$\Rightarrow Q \leq gPg^{-1}$ some g.

<u>SO</u> if $Q \in Syl_p(G)$ then $Q = gPg^{-1}$ some $g \in G$.
(order $= p^a$)

(8)

(3) $\cancel{n_p = |G : N_G(P)|}$

$\Rightarrow$ $n_p = r \equiv 1 \pmod{p}$

$n_p = |G : N_G(P)| \,\big|\, |G|$

QED

4.5

Applications of Sylow's Theorems

Examples

(1) If $p \nmid |G|$ then 1 is the unique Sylow p-subgroup
If $|G| = p^\alpha$ then G is the unique Syl p-subgr

(2) If G is abelian then there is a unique Sylow p-subgroup H_p of G for all primes p. H_p consists of all elements in G whose order is a power of p.
(p-primary component of G)

(3) $|S_3| = 6 = 2 \cdot 3$
Then $Syl_2(S_3) = \{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\}$
So $n_2(S_3) = 3$
and $Syl_3(S_3) = \{\langle(123)\rangle\}$, $n_3(S_3) = 1$

~~Since~~

(4) $|A_4| = 12 = 2^2 \cdot 3$
$Syl_2(A_4) = \{\langle(12)(34), (13)(24)\rangle\}$
$Syl_3(A_4) = \{\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle\}$

(5) $|S_4| = 24 = 2^3 \cdot 3$ D_8 is isomorphic to a subgroup of S_4. Hence every Sylow 2-subgr is isom to D_8.

Cauchy's Thm

Let G be a finite group, $p \mid |G|$ p prime. Then there exists an element of order p.

Pf Let P be a Sylow p-subgrp of G. Then $|P| = p^\alpha$, $\alpha > 0$

Let $g \in P$, $g \neq 1$. Then $|g| = p^\beta$ some β $0 < \beta \leq \alpha$.

Take $x = g^{p^{\beta-1}}$. Then $x^p = 1$

QED