MATH 403/503 L11

Burnside's Counting Formula

Q: When a group G acts on a set X, how many orbits are there?

Theorem Let G be a finite group acting on a finite set X. Then the number of orbits, $|G\backslash X|$, is given by Burnside's Formula:

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

where for each $g \in G$:

$$X^g = \{x \in X \mid g.x = x\}.$$

Proof Let

$$\Omega = \{ (g,x) \in G \times X \mid g.x = x \}$$

On the one hand,

$$|\Omega| = \sum_{g \in G} |X^g| \qquad (*)$$

On the other hand,

$$|\Omega| = \sum_{x \in X} |G_x|$$

Since $g G_x g^{-1} = G_{g.x}$ (by HW), and thus $|G_{g.x}| = |G_x|$, stabilizers from the same orbit have the same size. So we have

$$|\Omega| = \sum_{i=1}^{m} |\mathcal{O}_{x_i}| \cdot |G_{x_i}|$$

where $\{x_1, x_2, \dots, x_m\}$ is a set of representatives for the orbits.

By the orbit-Stabilizer Theorem,

$$|\mathcal{O}_{x_i}| \cdot |G_{x_i}| = |G|$$

So we get

$$|\Omega| = \sum_{i=1}^{m} |G| = m \cdot |G| \quad (**)$$

But $m = \#$ orbits $= |X \backslash G|$.

Combining $(*)$, $(**)$ we get

$$|X \backslash G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

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Ex How many ways can an equilateral triangle have sides painted, if 4 colors are allowed?

Sol. $G = D_3$ $\quad \triangle$

$$= \{e, r, r^2, s, sr, sr^2\}$$

$$X = \{\text{ways to paint } \triangle \text{ when fixed to plane}\}$$

$$G\backslash X = \{\text{ways to paint } \triangle \text{ up to symmetries}\}$$

$$|X^e| = |X| = 4 \times 4 \times 4 = 64$$

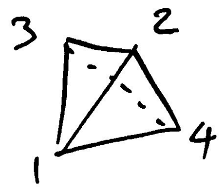
$$|X^r| = |X^{r^2}| = 4$$

$$|X^s| = |X^{sr}| = |X^{sr^2}| = 4 \cdot 4 = 16$$

So by Burnside's Formula:

$$|G\backslash X| = \frac{1}{6}(64 + 4 + 4 + 16 + 16 + 16) = \boxed{20}$$

Ex.



A_4

Paint a tetrahedron ~~in~~ using (at most) 3 colors.

$$|X^{(1)}| = 3\cdot 3\cdot 3\cdot 3 = 81 \qquad\qquad 1$$

$$|X^{(12)(34)}| = 3\cdot 3 \qquad (\times 3) \qquad 3$$

$$|X^{(123)}| = 3\cdot 3 \qquad (\times 4\cdot 2) \qquad \frac{8}{12}$$

$$|A_4 \backslash X| = \frac{1}{12}\left(81 + 3\cdot 9 + 8\cdot 9\right) =$$

$$= \frac{1}{4}\left(\underbrace{27 + 9}_{36} + 8\cdot 3\right) = 9 + 6 = \underline{\underline{15}}$$