Math 403/503 Spring 2024

Problems 4 and 5 were updated on March 4, 5:20pm.

In this class, by

- a *ring* R we mean a *ring with identity* 1_R;
- a *subring* $S \subseteq R$ we mean *subring containing the identity* 1_R;
- a *ring homomorphism* $\varphi : R \to T$ we mean *ring homomorphism sending* 1_R *to* 1_T.

Homework 5, due March 6

1. Determine which of the following sets are **rings** with respect to the usual operations of addition and multiplication. If the set is a ring, determine whether it is an **integral domain** or a **field** (or both). (*Hint:* Use the subring criterion where appropriate.)
   (a) $7\mathbb{Z}$
   (b) $\mathbb{Z}_{18}$
   (c) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$
   (d) $R = \{a + b\sqrt[3]{3} \mid a, b \in \mathbb{Q}\}$
   (e) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ where $i^2 = -1$
   (f) $\mathbb{Q}(\sqrt[3]{3}) = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} \mid a, b, c \in \mathbb{Q}\}$
2. Find all the units in the ring $M_2(\mathbb{Z}_2)$ of all 2×2-matrices with entries from $\mathbb{Z}_2$.
3. If R and S are rings, prove that $R \times S$ is a ring with component-wise operations.
4. Find all ring homomorphisms $\varphi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$ **or prove that there are none**.
5. Define a map $\varphi : \mathbb{C} \to M_2(\mathbb{R})$ by
$$\varphi(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
   Prove that φ is an **injective homomorphism** of rings.
6. Let $\varphi : R \to S$ be a ring homomorphism. Prove each of the following statements.
   (a) If R is a commutative ring, then $\varphi(R)$ is a commutative subring of S.
   (b) If R is a field, and $\varphi(R) \neq \{0_S\}$, then φ is injective.

7. Let R be a ring. Define the *center* of R to be

$$Z(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}.$$

Prove that $Z(R)$ is a commutative subring of R.