Math 403/503 Spring 2024

Homework 2, due January 31

1. Consider the abelian group $G = \mathbb{Z}_{12}$. Let $N = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ (where $\bar{a} = [a]_{12}$ denotes the congruence class of a modulo 12). Find the Cayley table for the group G/N. Compare it to the Cayley table for $\mathbb{Z}_3$.

2. Let $G = D_n = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$ be the dihedral group of order $2n$. Let $N = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. Show that N is a normal subgroup of G and that $G/N \cong \mathbb{Z}_2$.

3. Let G_1 and G_2 be groups with identity elements denoted e_1 and e_2 respectively. Let $G = G_1 \times G_2$.
   (a) Show that $\widetilde{G}_1 = \{(x, e_2) \mid x \in G_1\}$ is a normal subgroup of G.
   (b) Show that $G/\widetilde{G}_1$ is isomorphic to the group G_2. *Hint:* Find a surjective homomorphism $\varphi : G \to G_2$ with kernel equal to $\widetilde{G}_1$. Then use the First Isomorphism Theorem.

4. Show that $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. *Hint:* Consider the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_4$, $\varphi(a) = \big([a]_3, [a]_4\big)$.

5. Let M be a normal subgroup of a group G and let N be a normal subgroup of a group H. If $\varphi : G \to H$ is a homomorphism such that $\varphi(M) \subseteq N$, prove that the map $\psi : G/M \to H/N$ given by $\psi(gM) = \varphi(g)N$ is a well-defined homomorphism.