Math 403/503 Spring 2024

Practice Problems for Exam 2

1. (a) Prove that $x^5 - 9x + 3$ is irreducible over $\mathbb{Q}$.
   (b) Is $f(x) = x^4 + 2$ irreducible over $\mathbb{Q}$? Over $\mathbb{R}$?
   (c) Prove that $x^4 - 3x + 1$ is irreducible over $\mathbb{Q}$.

2. Let D be an integral domain and $P \subseteq D$ a prime ideal. Let $F = \operatorname{Frac} D$ be the field of fractions of D. Consider
$$D_P = \{\frac{a}{b} \mid b \notin P\}$$
   (a) Show that D_P is a subring of F.
   (b) Show that $\overline{P} := \{\frac{a}{b} \in D_P \mid a \in P\}$ is the unique maximal ideal of D_P.

3. Show that the composition of two ring homomorphisms is a ring homomorphism. What is the kernel of the composition?

4. Let R be a commutative ring and I be an ideal of $R[x]$. Let
$$I_0 = \{f(0) \mid f(x) \in I\}$$
be the set of constant terms of elements of I. Show that I_0 is an ideal of R and that
$$\frac{R[x]}{I + (x)} \cong \frac{R}{I_0}.$$
Hint: Consider an evaluation homomorphism $R[x] \to R$, composed with the canonical projection $R \to R/I_0$. Use the previous problem and the First Isomorphism Theorem for rings.

5. Let R and S be commutative rings and $\varphi : R \to S$ be a ring homomorphism. Let Q be a prime ideal of S. Prove that $\varphi^{-1}(Q)$ is a prime ideal of R.

6. Let $S = \mathbb{Z}[i]$ be the ring of Gaussian integers. Prove that $S[x]$ is a unique factorization domain.

7. If F is a field, prove that $F[x, y]$ is not a Euclidean domain.

8. If K and L are fields, show that $K \times L$ has exactly three maximal ideals.

9. Let R be a ring and $\varphi : R \to R$ be a ring automorphism (i.e. bijective ring homomorphism). Let I be an ideal of R.
   (a) Show that $\varphi(I)$ is also an ideal of R.
   (b) Prove that $\varphi(I)$ is prime iff I is prime.

10. Let P and Q be distinct nonzero prime ideal of a PID, R. Prove that
$$\varphi : R \longrightarrow \frac{R}{P} \times \frac{R}{Q}, \quad \varphi(r) = (r + P, r + Q)$$
is a surjective ring homomorphism.

11. Find a commutative ring R such that $R[x]$ contains a nonconstant unit.

12. Let F be a field. Let $R = F[x, y] = F[x][y]$ be the ring of polynomials in two indeterminates, and I be the principal ideal $(x^2 + y)$ of R. Prove that R/I is isomorphic to the polynomial ring $F[z]$.