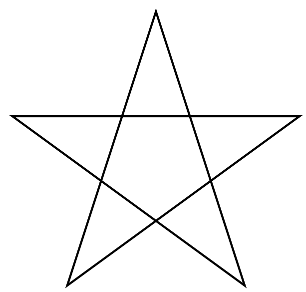
Math 403/503 Spring 2024

Practice Problems for Exam 1

1. If five colors are available, in how many ways can you paint the six regions of the following star shape (up to of rotations and reflections)? Each region must be painted in exactly one of the colors.



2. Determine which of the following permutations are conjugate in S_5:

$$(142)(35), \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$$

$$(354)(23), \qquad (14)(52), \qquad (152)(124)$$

3. Let G be a group, H a subgroup of G, and N a normal subgroup of G contained in H. Suppose H/N is a normal subgroup of G/N. Show that H is a normal subgroup of G.

4. Define $\varphi : \mathbb{Z} \to \mathbb{Z}_4 \times \mathbb{Z}_6$ by $\varphi(a) = ([a]_4, [a]_6)$. Find the kernel and image of φ. What does the First Isomorphism Theorem tell you in this case?

5. Let G be a group and H a subgroup of G. Show that the following defines a group action of $H \times H$ on G:

$$(h, k).g = hgk^{-1}, \qquad \text{for all } (h, k) \in H \times H \text{ and all } g \in G.$$

6. Let G be a group and X be a set. Let $\varphi : G \to S_X$ be a homomorphism. Show that the following formula defines an action of G on X:

$$g.x = \varphi(g)(x), \qquad g \in G,\ x \in X.$$

7. Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$ and $HK = G$. Show that the map $\psi : H \times K \to G$, $\psi(h,k) = hk$ is an isomorphism.

8. Let H and K be subgroups of a group G such that $H \leq N_G(K)$. Show that HK is a subgroup of G, and $K \trianglelefteq HK$.

9. Let G be a finite group and p a prime number, such that $g^p = e$ for all $g \in G$. Show that G has order p^α for some non-negative integer α.

10. Show that no group of order 96 is simple.

11. Show that every group of order 45 has a normal subgroup of order 9.

12. Let G be a group such that $G/Z(G)$ is cyclic. Show that G is abelian.

13. List all abelian groups of order 200, up to isomorphism.

14. How many abelian groups of order 144 are there up to isomorphism?

15. Show that the number of abelian groups of order 256 is equal to the number of congruence classes in S_8.